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A fiber approach to harmonic analysis of unfolded higher-spin field equations

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Abstract: In Vasiliev's unfolded formulation of higher-spin dynamics the standard fields are embedded on-shell into covariantly constant master fields valued in Lorentz-covariant slices of the star-product algebra $\mathcal{A}$ of functions on the singleton phase space. Correspondingly, the harmonic expansion is taken over compact slices of $\mathcal{A}$ that are unitarizable in a rescaled trace-norm rather than the standard Killing norm. Motivated by the higher-derivative nature of the theory, we examine indecomposable unitarizable Harish-Chandra modules consisting of standard massless particles plus linearized runaway solutions. This extension arises naturally in the above fiber approach upon realizing compact-weight states as non-polynomial analytic functions in $\mathcal{A}$.

Keywords: Gauge Symmetry, Field Theories in Higher Dimensions, Space-Time Symmetries.

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Contents

1. Introduction

1.1 Masslessness and compositeness

Free massless spin-s particles in four space-time dimensions fall into infinite-dimensional representations $\mathfrak{D}(s+1;(s,s))$ of the conformal group $\mathfrak{so}(4,2)$ with restrictions $\mathfrak{D}(s+1;(s,s))|_{\mathfrak{iso}(3,1)} \simeq \mathfrak{D}(m^2=0;|\lambda|=s)$ and $\mathfrak{D}(s+1;(s,s))|_{\mathfrak{so}(3,2)} \simeq \mathfrak{D}(s+1;(s))$. Conformal symmetry may be broken by Lorentz invariant interactions down to $\mathfrak{iso}(3,1)$, $\mathfrak{so}(3,2)$ or $\mathfrak{so}(4,1)$ depending on the cosmological constant Λ. One should distinguish between ordinary and exotic gauge theories depending on whether the infrared cutoff set by Λ is immaterial for local interactions or not, and adopt a unified approach starting with non-vanishing Λ [1]. If the (effective) field theory is ordinary then Λ may as well be kept finite as long as one describes processes at energy scales much larger than $\sqrt{|\Lambda|}$, which is where quantum field theory has been tested so far. At any such local rate, massless particles in four space-time dimensions exhibit compositeness: the lowest-energy representations $\mathfrak{D}(s+1;(s))|_{\mathfrak{so}(3,2)}$ are squares of a more fundamental representation namely Dirac's supersingleton $\mathfrak{D}(\frac{1}{2};(0))\oplus\mathfrak{D}(1;(\frac{1}{2}))$ [2], as summarized by the remarkable Flato-Fronsdal formula [3]:

$$\left[\mathfrak{D}\left(\frac{1}{2};(0)\right)\oplus\mathfrak{D}\left(1;\left(\frac{1}{2}\right)\right)\right]^{\otimes 2} = \bigoplus_{2s=0,1,2,\dots}\left[\mathfrak{D}(s+1;(s))\oplus\mathfrak{D}(s+1+\delta_{s0};(s))\right] \ . \tag{1.1}$$

Strictly speaking, the generators of $\mathfrak{so}(4,2)/\mathfrak{so}(3,2)$, and hence $\mathfrak{so}(4,1)$, act on the left-hand side in a non-tensorial split (see comment in appendix G.2) and $\mathfrak{D}(m^2=0;|\lambda|=s)|_{\mathfrak{iso}(3,1)}$ is recovered at scales much larger than $|\Lambda|$ [4], as mentioned above.

Compositeness is directly related to the higher-spin Lie-algebra extension $\mathfrak{ho}(D+1;\mathbb{C})$ of $\mathfrak{so}(D+1;\mathbb{C})$ (see, for example, [5−7] and references therein) that: i) acts transitively on the D-dimensional scalar singleton $\mathfrak{D}(\epsilon_0;(0))$, $\epsilon_0=\frac{1}{2}(D-3)$, which thus functions as its fundamental lowest-energy representation; and ii) arranges the D-dimensional Flato-Fronsdal tower $[\mathfrak{D}(\epsilon_0;(0)))]^{\otimes 2}=\bigoplus_{s=0}^{\infty}\mathfrak{D}(s+2\epsilon_0;(s))$ into an irrep which one may refer to as the massless higher-spin multiplet.[1] As we shall show below (see section 3.7), the adjoint representation itself is also composite and isomorphic to the tensor product of a singleton (lowest-energy representation) and its negative energy counterpart, an anti-singleton (highest-energy representation):

$$\mathfrak{ho}(3,2)\simeq\left[\mathfrak{D}^+\left(\frac{1}{2};(0)\right)\otimes\mathfrak{D}^-\left(-\frac{1}{2};(0)\right)\right]\oplus\left[\mathfrak{D}^-\left(-\frac{1}{2};(0)\right)\otimes\mathfrak{D}^+\left(\frac{1}{2};(0)\right)\right]. \quad (1.2)$$

There appears to be a clear separation between interactions that break and preserve higher-spin gauge symmetry. In particular, the latter are exotic in a sense to be discussed next.

1.2 Interlude: canonical vs unfolded higher-spin equations

In the standard approach to field theory, based on kinetic terms built using the metric, perturbative higher-spin gauge interactions necessarily contain higher derivatives. Removing unnecessary derivatives by perturbative field redefinitions leads to standard vertices with minimal required numbers of derivatives, that one may refer to as the canonical vertices. The number of derivatives in the canonical spin $s_1-s_2-s_3$ vertices grows linearly with $s_1+s_2+s_3$ [14−18]. For $\Lambda=0$ these vertices are ordinary and hence weakly coupled at sufficiently low energies. In particular, in the case of $s_1=2$ and $s_2=s_3=s>2$ there are minimal non-abelian vertices with $2s-2$ derivatives, Chern-Simons-like vertices (albeit parity preserving) with $2s$ derivatives, and Born-Infeld-like vertices with $2s+2$ derivatives. However, the standard gravitational cubic spin $2-s-s$ coupling (with two derivatives) is plagued by a classical anomaly, and hence the non-linear completion based on non-abelian higher-spin symmetries seems problematic in an expansion around flat spacetime (although some of the above-mentioned cubic vertices do pass certain consistency tests at the quartic level [19]).

On the other hand, if $\Lambda\neq 0$ there exists an exotic cancelation mechanism found by Fradkin and Vasiliev [16], whereby the "non-gravitational" minimal spin $2-s-s$ vertex from flat space, which has $2s-2$ derivatives, is completed with lower derivatives all the way down to the standard gravitational two-derivative coupling, with dimensionful parameters fixed in units of Λ. Moreover, there exists a fully non-linear completion in the form of Vasiliev's

[1] By definition, a massless lowest-weight space of $\mathfrak{so}(D+1;\mathbb{C})$ results from factoring out a non-trivial ideal, corresponding to the longitudinal modes of a gauge field, from a generalized Verma module (see appendix A). For $D>4$ one needs to distinguish between conformal, composite and partial masslessness [8−11] (see also [12, 13] and references therein). In this paper we focus on composite masslessness.

unfolded master-field equations ([20, 21], see also [22, 23] for reviews and more references) which are manifestly higher-spin gauge invariant and locally homotopy invariant, i.e. referring to the base manifold only via differential forms and the exterior derivative d, as well as manifestly background independent in the sense that their salient features do not rely on the expansion around any specific solution nor on the reference to a (non-degenerate) metric. The equations actually assume a remarkably simple form: they describe a covariantly constant zero-form and a flat connection taking their values in a deformed "fiber". The crux of the matter lies, of course, in what constitutes a good set of observables.

In the sub-sector where the Weyl zero-form is "weak",[2] the master-field equations imply generally covariant and locally Lorentz invariant, albeit non-canonical, standard field equations for perturbatively defined dynamical components of the master fields, that one may refer to as the microscopic fields. In particular, there appear a microscopic vielbein and spin-connection defining what one may refer to as the Vasiliev frame, to be related to the canonical Einstein frame by a (highly non-local) field redefinition. The microscopic standard field equations (with box-like kinetic terms) are obtained by eliminating auxiliary fields after first having expanded the master-field equations in weak curvatures and then around "large" Vasiliev-frame metric while treating the remaining microscopic fields as "small". The result contains two parameters:

(i) a dimensionless AdS-Planck constant $g^2 \equiv (\lambda \ell_p)^{D-2}$ that counts the order in the perturbative weak-field expansion, where ℓ_p enters via the normalization of the effective standard action and we are working with dimensionless physical fields; and

(ii) a massive parameter λ that simultaneously

(iia) sets the infrared cutoff via $\Lambda \sim \lambda^2$ and critical masses $M^2 \sim \lambda^2$ for the dynamical fields; and

(iib) dresses the derivatives in the interaction vertices thus enabling the Fradkin-Vasiliev (FV) mechanism.

At fixed order in g, that one may always take to be a small parameter, the non-canonical microscopic interactions are given by Born-Infeld-like series expansions in $\lambda^{-2}\nabla\nabla$, that one may refer to as tails (see [26] for a computation of the quadratic scalar-field contributions to the microscopic stress tensor). On general grounds, these microscopic tails should be related to the canonical vertices via non-local, potentially divergent, field redefinitions. Thus one has the following scheme:

$$\begin{array}{ccccc} \text{Unfolded} & \scriptstyle\text{weak} & \text{Standard-exotic} & \scriptstyle\text{non}-\text{local} & \text{Standard-exotic} \\ \text{master-field} & \overset{\scriptstyle\text{fields}}{\leftrightarrows} & \text{microscopic} & \overset{\scriptstyle\text{field redef.}}{\rightleftarrows} & \text{canonical} \\ \text{equations} & & \text{field equations} & & \text{field equations} \end{array} \tag{1.3}$$

We stress that what makes higher-spin theory exotic is the dual purpose served by λ within the FV mechanism whereby positive and negative powers of λ appear in mass terms

[2]This notion can be formulated in more precise terms using a set of local zero-form observables which are given by traces of algebraic powers of the zero-form [24, 25].

JHEP10(2008)022

and vertices,[3] respectively. Thus, at each order of the canonical expansion scheme, the local bulk interactions — and in particular the standard minimal gravitational two-derivative couplings — are dominated by strongly coupled "top vertices" going like finite positive powers of (energy scale)/(IR cutoff). On the other hand, in the microscopic expansion scheme each order is given by a potentially divergent Born-Infeld tail, suggesting that classical solutions as well as amplitudes should be evaluated directly within the master-field formalism, which offers transparent methods based on requiring associativity of the operator algebra for setting up and assessing regularized calculational schemes.

We also emphasize that the metric-dependent standard symplectic structure differs from the background-independent unfolded symplectic structure, which treats all the derivatives of the physical fields as a priori independent variables [28]. Besides providing a more systematic handling of the initial data formulation in the presence of higher derivatives, the unfolded structure leads to a Gaussian path-integral measure that suppresses higher-derivative fluctuations.

The above-mentioned issues of regularization and off-shell formulation go beyond the present scope of this paper, although they serve as key motivations for our work. Similar considerations are also made in reference [18].

1.3 Phase-space quantization

The higher-spin master-field formalism of [29, 30] has been conjectured in [7] to be directly connected to the geometric Cattaneo-Felder formulation [31] of the phase-space quantization [32] of singletons in terms of a two-dimensional first-order parent action, whose BV quantization leads to an embedding of the algebra $\mathcal{A}$ of functions on the singleton phase space into an infinite-dimensional quantum version $\widehat{\Gamma}$ governed by a "topological" BRST operator $\widehat{q}$. One is led to attempt to formulate a complete theory with:

(i) a first-order total Lagrangian for locally defined master fields with "kinetic" terms built from $d+\widehat{q}$;

(ii) globally defined observables that are given by integrals over (cycles in) the space-time base manifold times (cycles in) the quantum phase space;

(iii) two coupling constants g and λ^{-1} corresponding to the perturbative expansions of the base manifold and of the internal sigma model,[4] respectively; and

(iv) a limit in which the master fields are classical differential forms on the base manifold taking their values in $\widehat{\Gamma}$ (c.f. classical string-field theory).

[3] This brings to mind the transition from the standard perturbative regime of closed string theory in 10D flat space where physical, string and Planck lengths ℓ, ℓ_s and $\ell_p = g_s^{\frac{1}{4}}\ell_s$ obey $\ell_p \ll \ell_s \ll \ell$, to a high-energy regime where $\ell_p \ll \ell \ll \ell_s$, and thus ℓ_s switches role from a stringy UV cutoff to a stringy IR cutoff (e.g. in one-loop amplitudes where the fundamental domain can be chosen such that $\ell/\ell_s, \mathrm{Im}\tau \in [0,1]$). Sending ℓ_s to infinity, which in string-field language means taking $\langle e^{\phi_{\rm dil}}\rangle = g_s \to 0$ keeping ℓ_p fixed, may lead to an unbroken phase with stringy IR cutoff set by λ [27].

[4] The flat and infinitely curved limits $\lambda \to 0$ and $\lambda \to \infty$ should thus correspond to the strongly and weakly coupled limits of the sigma model, respectively.

Vasiliev's fiber algebra, that we shall denote by $\widehat{\mathcal{A}}$, arises as a truncation of $\widehat{\Gamma}$ [7] that is formally associative, and equipped with trace operations given by integrations over suitable cycles in $\widehat{\mathcal{A}}$ [7, 24, 33]. Thus Vasiliev's full master fields take their values in $\widehat{\mathcal{A}}$, and the simply-looking full unfolded master-field equations makes use only of the ordinary associative $\star$-product, or 2-product, on $\widehat{\mathcal{A}}$. This algebra contains $\mathcal{A}$ as a subalgebra. Correspondingly, there are reduced master fields taking their values in $\mathcal{A}$ and obeying highly non-linear reduced unfolded master-field equations making use of n-products with $n = 2, 3, \ldots$ given perturbatively in the weak-field expansion.

These equations admit a free limit on maximally symmetric spaces, such that the properties of free standard higher-spin fields are "dual" to properties of $\mathcal{A}$ viewed as an $\mathfrak{ho}(D+1;\mathbb{C})$-module. As such $\mathcal{A}$ is isomorphic to the enveloping algebra $\mathcal{U}[\mathfrak{so}(D+1;\mathbb{C})]$ modulo the ideal consisting of all elements that are trivial in the scalar singleton module. The ideal is generated by the covariant scalar singleton equations of motion[5] [8] (see section 2 for notation):

$$\frac{1}{2} M_{\{A}{}^{C} \star M_{B\}C} \approx 0 , \qquad M_{[AB} \star M_{CD]} \approx 0 . \tag{1.4}$$

In [8] it was found that the first condition, when imposed on lowest-weight spaces, selects the singletons, further restricted to be scalars (or spinors, in $D = 4$) by the second condition. Moreover, somehow in the spirit of the deformation quantization program [32], in the following we shall exploit the fact that the standard one-particle states can be mapped to non-polynomial Weyl-ordered phase-space functions corresponding to similar special functions in the enveloping-algebra realization of $\mathcal{A}$. For example, in $D = 4$ we have

$$\begin{array}{ccccc} \text{Free one-particle} & & \text{Phase-space} & & \text{Enveloping-algebra} \\ \text{lowest-weight state} & \leftrightarrows & \text{function} & \leftrightarrows & \text{element} \\ |1;(0)\rangle = |\tfrac{1}{2};(0)\rangle \otimes |\tfrac{1}{2};(0)\rangle & & 4[\exp(-2\bar{a}^i a_i)]_{\rm Weyl} & & 4\exp(-4E) \; . \end{array} \tag{1.5}$$

The incorporation of lowest-weight and highest-weight spaces into $\mathcal{A}$ is a manifestation of compositeness (and it applies equally well to finite-dimensional lowest-and-highest-weight spaces). However, the enveloping-algebra approach, which does not refer a priori to lowest-weight states, is more general, and it indeed gives rise to additional representations, some of which do not contain any lowest nor highest-weight state. We stress that, while these representations arise in a dual fashion on the standard free field-theory side, we are mainly interested in the enveloping-algebra side which seems more relevant for computing the exotic higher-spin interactions, as discussed above. We shall explore the possibility of using such a fiber approach to naturally incorporate the massless modules $\mathfrak{D}(s+1;(s))$ and linearized runaway solutions of higher-spin gauge theories in a bigger, indecomposable module.

1.4 Runaway solutions and extended compositeness

In an ordinary field theory one may consider three distinct classes of solutions: particles, runaway solutions and solitons. For a free massless scalar field in flat spacetime one has:

[5] An affine extension of the singleton equations of motion, related to discretized tensionless p-branes in AdS, has been examined in [34] at the level of lowest-weight modules.

(i) positive and negative energy mass shells $\mathfrak{D}^{\pm}(0;0)$ with continuous spectrum for the translation generator $P_a = (P_0, P_r)$, with $a = 0, 1, \ldots, D-1$;

(ii) the $\mathfrak{iso}(D-1,1)$ orbit $\mathfrak{W}(\ell)$ of the static runaway spin-ℓ solution $r^\ell Y_\ell(\widehat{n})$, $\ell = 0, 1, \ldots,$ where r is the radial coordinate and Y_ℓ are spherical harmonics obeying $(\nabla^2_{|S^{D-2}} + \ell(\ell+2\epsilon_0))Y_\ell = 0$ with $\epsilon_0 = \frac{1}{2}(D-3)$; and

(iii) the $\mathfrak{iso}(D-1,1)$ orbit $\widetilde{\mathfrak{W}}(\ell)$ of the static singular spin-ℓ solution $r^{-\ell-2\epsilon_0} Y_\ell(\widehat{n})$.

The mass shells can be given in a basis $\{|e;(\ell)\rangle\}$, with $e \in \mathbb{R}_+$ and $\ell = 0, 1, 2, \ldots$, where each energy level forms a finite-dimensional irrep of the $\mathfrak{so}(2)$ generated by P_0 and the $\mathfrak{so}(D-1)$ generated by the spatial rotations M_{rs}, which one refers to as an $(\mathfrak{so}(2) \oplus \mathfrak{so}(D-1))$-type. In this basis the plane-waves with energy $e \neq 0$ are given by $|(e, e\hat{p}_r)\rangle = \sum_\ell \frac{(ie)^\ell}{\ell!} c_\ell |e;(\ell)\rangle$ for coefficients c_ℓ that are non-vanishing for all ℓ and with the $e = 0$ limit $|(0,0)\rangle = |0;(0)\rangle$, while the static spin-$\ell$ runaways are given by $|0;(\ell)\rangle$. The set of runaway modules $\{\mathfrak{W}(\ell)\}$, which are simply the spaces of traceless polynomials of degree ℓ in the cartesian coordinates x^a, forms an indecomposable chain (see appendix A) in which $\mathfrak{W}(\ell) \subset \mathfrak{W}(\ell+1)$, and $\{\widetilde{\mathfrak{W}}(\ell)\}$ exhibits the "dual" structure $\widetilde{\mathfrak{W}}(\ell) \supset \widetilde{\mathfrak{W}}(\ell+1)$. Massless static fields ϕ have conformal weight ϵ_0 in the $D-1$ spatial dimensions, and consequently the runaway and singular spin-ℓ solutions are related by

$$\begin{array}{c} \text{runaway} \\ \phi_\ell = r^\ell Y_\ell(\hat{n}) \end{array} \quad \overset{r \leftrightarrow \frac{1}{r}}{\rightleftarrows} \quad \begin{array}{c} \text{singular} \\ \widetilde{\phi}_\ell = r^{-\ell-2\epsilon_0} Y_\ell(\hat{n}) \end{array} . \tag{1.6}$$

One also notes that, whereas $\mathfrak{D}(0;(0)) \simeq \mathfrak{D}'(0;(0))$ with $\mathfrak{D}'(e;(0))$ being the compact weight-space module consisting of states $|\psi\rangle \in \mathfrak{D}(0;(0))$ of the form $|\psi\rangle = \int_{e>0} de\, \psi(e)\, |e;(\ell)\rangle$, the action of P_0 cannot be diagonalized in $\mathfrak{W}(\ell)$ and $\widetilde{\mathfrak{W}}(\ell)$ which can therefore not be decomposed in terms of $(\mathfrak{so}(2) \oplus \mathfrak{so}(D-1))$-types.

The effects of Λ and/or mass terms show up in the large-r/infrared limit, while they do not alter the small-r/ultraviolet limit. For $\Lambda < 0$ and unitary masses, the free-field fluctuations that are normalizable in the Killing norm $\|\cdot\|_{\rm can}$, induced from the standard canonical action, form a lowest-weight representation $\mathfrak{D}$ containing the one-particle states [35]. Among the linearized fields with divergent $\|\cdot\|_{\rm can}$ are the runaway and singular solutions. These may have non-linear completions depending on the details of the interactions. In ordinary field theories some of the singular solutions may become solitonic objects with finite energy and localizable energy density. In exotic field theories, on the other hand, one may expect that the perturbative descriptions of localized soliton (or particle) solutions involve derivative expansions that are coupled more strongly than those of corresponding runaway solutions. One might entertain the idea that the Born-Infeld tails of higher-spin gauge theory lead to a restoration of the duality (1.6) at the full level. Another noteworthy feature of higher-spin gauge theory is that all even one-particle and runaway $(\mathfrak{so}(2) \oplus \mathfrak{so}(D-1))$-types, of all fields, are generated by the $\mathfrak{ho}(D-1,2)$ action from the static runaway mode $\phi_{0;(0)}$ of the scalar field obeying $(\nabla^2 + 4\epsilon_0)\phi = 0$; for example in AdS_4 with metric $ds^2 = \frac{1}{\cos^2\xi}(-dt^2 + d\xi^2 + \sin^2\xi d\Omega^2_{S^2})$ one has

$$\begin{array}{ccc} \text{Static, } \ell = 0 & & \text{Static, } \ell = 0 \\ \text{runaway field} & \overset{\text{unfolding}}{\leftrightarrows} & \text{analytic } \mathcal{A} \text{ function} \\ \phi_{0;(0)}(\xi) \ = \ \frac{\xi}{\tan\xi} & & \frac{\sinh 4E}{4E} \end{array} . \tag{1.7}$$

The above reasoning and the fashion in which the degeneracies at the infra-red "apices" of the flat-space massless mass-shells are resolved by the deformation by $\Lambda < 0$, whereby particles and runaway solutions fall into distinct $\mathfrak{so}(2) \oplus \mathfrak{so}(D-1)$-types, suggest a unification in higher-spin gauge theory of the one-particle states in $\mathfrak{D}$ and the runaway solutions $\mathcal{W}$ into a single module $\mathcal{M}$ that:

(i) has the indecomposable structure $\mathcal{M} = \mathfrak{D} \in\!\!\!\!+\ \mathcal{W}$ (runaway solutions are allowed to emit particles into the bulk), where the definition of the semi-direct sum symbol $\in\!\!\!\!+$ is recalled in appendix A;

(ii) is factorizable, i.e. realized via special functions in $\mathcal{A}$;

(iii) is generated by the $\mathfrak{so}(D-1,2)$ action starting from the D-dimensional generalization of the 4D static ground state (1.7);

(iv) is unitarizable (so that the duality (1.6) may make sense); and

(v) has an associative structure that lifts to $\widehat{\mathcal{A}}$ (existence of perturbatively defined full master fields).

In this paper we shall focus on (i)–(iv), leaving (v) for a future publication [36].

2. Enveloping-algebra realization of higher-spin master fields

The unfolded formulation of minimal bosonic higher-spin gauge theory in $D \geqslant 4$ spacetime dimensions makes use of two (reduced) master fields: a one-form A and a zero-form Φ taking their values, respectively, in the adjoint and twisted-adjoint representations $\mathfrak{ho}_0$ and $\mathcal{T}_+$ of the minimal higher-spin extension of $\mathfrak{so}(D+1;\mathbb{C})$. In this section, we describe how these representations arise in the associative algebra $\mathcal{A}$ obtained by factoring out the singleton annihilator $\mathcal{I}[V]$ from the universal enveloping algebra $\mathcal{U}[\mathfrak{so}(D+1;\mathbb{C})]$. The ideal consists of monomials containing at least two contracted or three anti-symmetrized $(D+1)$-dimensional vector indices, so that $\mathcal{A}$ consists of rectangular $\mathfrak{so}(D+1;\mathbb{C})$ tensors of height two. The adjoint and twisted-adjoint $\mathfrak{ho}_0$ actions are induced, respectively, via the commutator and a twisted version that represents the Lorentz translations P_a by the anti-commutator $\{P_a, \cdot\}$. As a result, $\mathfrak{ho}_0$ and $\mathcal{T}_+$ decompose under $\mathfrak{so}(D+1;\mathbb{C})$ into irreducible levels containing, respectively, the finite sets of gauge fields and infinite sets of Weyl-tensor/matter-field zero-forms required within the unfolded formalism for describing one spin-s degree of freedom (where s is related to the level).

In the leading order of the weak-field expansion, the zero-forms obey a covariant constancy condition and source the one-forms via a Chevalley-Eilenberg cocycle of the form

JHEP10(2008)022

$F_{(1)} = e^a \wedge e^b J_{(1)ab}$, where $F_{(1)}$ is the linearized adjoint curvature two-form, e^a is the vielbein, and $J_{(1)ab}$ is linear in Φ. Thus, the local propagating degrees of freedom are carried by Φ while A may carry zero-modes (that are solutions with $\Phi = 0$ that cannot be gauged away) [37]. As we shall see, the cocycle has a weight-space analog in the form of a short exact sequence connecting the adjoint levels to the corresponding composite massless lowest-weight spaces.

Next, we define a nontrivial trace operation $\mathrm{Tr}_{\mathcal{A}}$ on the algebra $\mathcal{A}$ that will endow the latter with a nondegenerate inner product, and related *reflector* states that possess a series of useful properties. These tools will be useful for establishing the state/operator correspondence mentioned in the Introduction. For example, as we shall examine later, the reflectors enable presentations of the master fields as elements in left bimodules. The definition of $\mathrm{Tr}_{\mathcal{A}}$ generalizes the supertrace operation first given in [33] for the case of the 4D oscillator realization of higher-spin superalgebras, as shown in appendix G.2.

Finally, we shall define harmonic expansions of Φ in maximally symmetric backgrounds as maps between the Lorentz-covariant and maximally compact slicings of the infinite-dimensional twisted-adjoint modules. This formalism will then be examined in greater detail in the cases of de Sitter and anti-de Sitter geometry.

2.1 Associative quotient algebra

The universal enveloping algebra $\mathcal{U}[\mathfrak{g}]$ of a Lie algebra $\mathfrak{g}$ is the associative algebra consisting of the unity $\mathbb{1}$ and arbitrary polynomials in the generators M_α of $\mathfrak{g}$ modulo the commutation rules, viz.[6] $\mathcal{U}[\mathfrak{g}] = \bigoplus_{n=0}^{\infty} \mathcal{U}_n$, $\quad \mathcal{U}_n \ni X_n = x^{\alpha(n)} M_{\alpha(n)}$, where $x^{\alpha(n)} \in \mathbb{C}$ and the basis consists of the symmetrized monomials

$$M_{\alpha(n)} = M_{\alpha_1} \cdots M_{\alpha_n} \equiv \begin{cases} \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} M_{\alpha_{\pi(1)}} \star \cdots \star M_{\alpha_{\pi(n)}} & \text{for } n = 1, 2, \ldots, \\ \mathbb{1} & \text{for } n = 0 \end{cases} , \quad (2.1)$$

where $\star$ and juxtaposition, respectively, denote the non-commutative product of $\mathcal{U}[\mathfrak{g}]$ and the commutative product given by symmetrization of the M_α, so that $X_m \star X_n = X_m X_n + \sum_{p \leqslant m+n-1} X'_p$. The algebra $\mathcal{U}[\mathfrak{g}]$ has the involutive anti-automorphism[7] given by the "transposition"

$$\tau(M_\alpha) = -M_\alpha \, , \qquad \tau(X_n) = (-1)^n X_n \, . \quad (2.2)$$

The adjoint and anti-commutator actions of $\mathcal{U}[\mathfrak{g}]$ on itself defined by $\mathrm{ad}_X(Y) = [X, Y]_\star$ and $\mathrm{ac}_X(Y) = \{X, Y\}_\star$ obey the closure relation $[\mathrm{ac}_X, \mathrm{ac}_Y]_\star = \mathrm{ad}_{[X,Y]_\star}$.

The Lie algebra $\mathfrak{g} = \mathfrak{so}(D+1; \mathbb{C})$ has generators M_{AB} obeying

$$[M_{AB}, M_{CD}]_\star = M_{AB} \star M_{CD} - M_{CD} \star M_{AB} = 4i\eta_{[C|[B} M_{A]|D]} \, , \quad (2.3)$$

where $A = (\sharp, a)$, $a = (\sharp', r)$, $r = 1, \ldots, D-1$, and $\eta_{AB} = \mathrm{diag}(-\sigma, \eta_{ab})$, $\eta_{ab} = \mathrm{diag}(-\sigma', \delta_{rs})$ with $\sigma, \sigma' = \pm 1$. We write $\sharp = 0'$ and $\sharp = D+1$ when $\sigma = +1$ and $\sigma = -1$, respectively,

[6] The symmetrized monomials are linearly independent according to the Poincaré-Birkhoff-Witt theorem.

[7] A map ϕ from an associative algebra to itself is an automorphism if $\phi(X \star Y) = \phi(X) \star \phi(Y)$ and an anti-automorphism if $\phi(X \star Y) = \phi(Y) \star \phi(X)$. An (anti)-automorphism is said to be involutive if $\phi^2 = 1$.

and $\sharp' = 0$ and $\sharp' = D$ when $\sigma' = +1$ and $\sigma' = -1$, respectively, and refer to $\sigma' = 1$ and $\sigma' = -1$, respectively, as Lorentzian and Euclidean signatures. Splitting $M_{AB} \to (M_{ab}, P_a)$, $P_a = M_{\sharp a}$, the commutation rules read

$$[M_{ab}, M_{cd}]_\star = 4i\eta_{[c|[b}M_{a]|d]}\,, \qquad [M_{ab}, P_c]_\star = 2i\eta_{c[b}P_{a]}\,, \qquad [P_a, P_b]_\star = i\sigma M_{ab}\,. \quad (2.4)$$

We let $\mathfrak{m} \simeq \mathfrak{so}(D;\mathbb{C})$ and $\mathfrak{s} \simeq \mathfrak{so}(D-1;\mathbb{C})$ denote the subalgebras generated by M_{ab} and M_{rs}, respectively, and refer to them as the Lorentz and spin algebras. The energy operator E is defined by $[E, \mathfrak{s}] = 0$ and we write $\mathfrak{e} = \mathbb{C} \otimes E$. The real forms $\mathfrak{so}(-\sigma, -\sigma', D-1)$, $\mathfrak{so}(-\sigma', D-1)$ and $\mathfrak{so}(D-1)$ of $\mathfrak{g}$, $\mathfrak{m}$ and $\mathfrak{s}$, respectively, are defined by $(M^{\mathbb{R}}_{AB})^\dagger = M^{\mathbb{R}}_{AB}$. The corresponding maximally symmetric spaces of radius λ^{-1} are

$$S(\sigma, \sigma') = \frac{\mathrm{SO}(-\sigma, -\sigma', D-1)}{\mathrm{SO}(-\sigma', D-1)} = \left\{X \in \mathbb{R}^{D+1} \quad \lambda^2 X^A X^B \eta_{AB} = -\sigma \right\}, \quad (2.5)$$

i.e., $S(+,+) = AdS_D$, $S(+,-) = H_D$, $S(-,+) = dS_D$ and $S(-,-) = S^D$. One splits $\mathfrak{so}(-\sigma, -\sigma', D-1) = \mathfrak{h} \oplus \mathfrak{l}$ where $\mathfrak{h} \simeq \mathfrak{so}(p) \oplus \mathfrak{so}(D+1-p)$ is the maximal compact subalgebra and $[\mathfrak{h}, \mathfrak{l}] = \mathfrak{l}$ and $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{h}$.

The monomials X_n with $n \geqslant 2$ decompose under $\mathrm{ad}_{\mathfrak{g}}$ into η_{AB}-traceless and Young-projected tensors. The trace parts and shapes with more than two rows form the ideal [34]

$$\mathcal{I}[V] = \left\{X = V \star X' \text{ for } X' \in \mathcal{U}[\mathfrak{g}]\right\}, \qquad V = \lambda^{AB} V_{AB} + \lambda^{ABCD} V_{ABCD}\,, \quad (2.6)$$

where $\lambda^{AB}, \lambda^{ABCD} \in \mathbb{C}$ and

$$V_{AB} \equiv \frac{1}{2} M_{(A}{}^C M_{B)C} - \frac{1}{2(D+1)} \eta_{AB} M^{CD} M_{CD}\,, \qquad V_{ABCD} \equiv M_{[AB} M_{CD]}\,, \quad (2.7)$$

absorbing the trace parts and more-than-two-row shapes, respectively. Defining $\mathcal{U}'[\mathfrak{g}] = \mathcal{U}[\mathfrak{g}] \setminus \mathbb{1}$, one has the chain $\mathcal{U}[\mathfrak{g}] \supset \mathcal{U}'[\mathfrak{g}] \supset \mathcal{I}[V]$ of proper ideals. Factoring out $\mathcal{I}[V]$ yields the infinite-dimensional unital associative algebra[8]

$$\mathcal{A} \equiv \frac{\mathcal{U}[\mathfrak{g}]}{\mathcal{I}[V]} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n\,, \qquad \mathcal{A}_n \ni X_n \approx x^{A(n),B(n)} M_{A(n),B(n)}\,, \quad (2.8)$$

where $X \approx X'$ means $X - X' \in \mathcal{I}[V]$. A basis for $\mathcal{A}$ consists of the traceless type-(n,n)

[8] Let $V = \lambda^{A_1,\dots,A_n} V_{A_1,\dots,A_n}$ with $\lambda^{A_1,\dots,A_n} \in \mathbb{C}$ and $V_{A_1,\dots,A_n}$ given by an $\mathfrak{g}$-irreducible, i.e. Young projected and traceless, monomial in M_{AB} and η_{AB}, thus obeying $[M_{BC}, V_{A_1,\dots,A_n}]_\star = 2i\sum_{i=1}^n \eta_{A_i[C} V_{\dots,A_{i-1},|B],A_{i+1},\dots,A_n}$. It follows that if $X \in \mathcal{U}[\mathfrak{g}]$ then $X \star V = V \star X'$ for some $X' \in \mathcal{U}[\mathfrak{g}]$. Thus $\mathcal{I}[V] \equiv V \star \mathcal{U}[\mathfrak{g}] = \mathcal{U}[\mathfrak{g}] \star V$ is a two-sided ideal and $[X \star V \star X'] = 0$ in $\mathcal{U}[\mathfrak{g}]/\mathcal{I}[V]$, that is $X \star V \star X' \approx 0$ for all $X, X' \in \mathcal{U}$. The above holds true also if V is $\mathfrak{g}$-reducible.

tensors[9]

$$\begin{aligned} M_{A(n),B(n)} &= M_{\{A_1B_1}\cdots M_{A_nB_n\}} \;=\; M_{\{A_1B_1}\star\cdots\star M_{A_nB_n\}} \\ &=\sum_{k=0}^{[n/2]}\kappa_{n;k}\eta_{\langle A_1A_2}\eta_{B_1B_2}\cdots\eta_{A_{2k-1}A_{2k}}\eta_{B_{2k-1}B_{2k}}M_{A_{2k+1}B_{2k+1}}\cdots M_{A_nB_n\rangle}\,, \end{aligned} \tag{2.9}$$

where $\kappa_{n;k}$ are fixed by $\eta^{CD}M_{A(n),B(n-2)CD}=0$ and $\kappa_{n;0}=1$, and the trace parts contain monomials of rectangular shapes. For example, if $n=2$ one has

$$M_{A(2),B(2)} = M_{A_1B_1}\star M_{A_2B_2}-\frac{2}{D(D+1)}(\eta_{A_1A_2}\eta_{B_1B_2}-\eta_{A_1B_1}\eta_{A_2B_2})C_2[\mathcal{S}]\,, \tag{2.10}$$

where $C_2[\mathcal{S}]$ is the quadratic Casimir operator $\frac{1}{2}M^{AB}\star M_{AB}$ given by (2.16) in the left $\mathcal{A}$-module $\mathcal{S}$ that we shall define below. For our purposes, we need

$$\mathrm{ad}_{M_{AB}}(M_{C(n),D(n)})=2in\left(\eta_{\{C_1|[B}M_{A]|C(n-1),D(n)\}}+\eta_{\{D_1|[B|}M_{|C(n)|,|A]|D(n-1)\}}\right)\,, \tag{2.11}$$

and (see also (A.26) and (B.16))

$$\mathrm{ac}_{M_{AB}}(M_{C(n),D(n)}) = 2\Delta_n M_{[A|\{C(n),|B]D(n)\}}+2\lambda_n\eta_{\{C_1|[A}\eta_{B]|D_1}M_{C(n-1),D(n-1)\}}\,, \tag{2.12}$$

where $\Delta_n=\frac{2(n+1)}{n+2}$ is fixed by the Young projection[10] while the contraction coefficient

$$\lambda_n = -\frac{1}{2}\frac{n(n+1)(n+\epsilon_0-1)}{n+\epsilon_0+\frac{1}{2}}\,, \tag{2.14}$$

can be computed by imposing either

(i) the trace condition using $V\approx 0$; or

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[9] We refer to tensors with the symmetry of the Young diagram with n_i cells in the ith row ($i=1,\ldots,\nu$) as type-$(n_1,\ldots,n_\nu)$ tensors, and work with normalized and mostly symmetric projectors

$$\mathbf{P}_{n_1,n_2,\ldots,n_\nu}=\frac{1}{\prod_{\text{cells}}(\text{hook-lengths})}\prod_{\text{rows }i}\mathbf{S}_i\prod_{\text{columns }j}\mathbf{A}_j\,,$$

where $\mathbf{S}_i$ and $\mathbf{A}_j$ are row symmetrizers and column anti-symmetrizers, respectively. Thus a type-$(n_1,\ldots,n_\nu)$ tensor has ν groups of symmetric indices $A^i(n_i)=A^i_1\ldots A^i_{n_i}$ subject to the over-symmetrization rule $M_{\cdots,(A^i_1\ldots A^i_{n_i},A^{i+1}_1)A^{i+1}_2\ldots A^{i+1}_{n_{i+1}},\cdots}=0$, $i=1,\ldots,\nu-1$. Young-projected index blocks are enclosed by $\langle\cdots\rangle$ and by $\{\cdots\}$ if they are also traceless. With the exception of (2.1), indices distinguished by subindexation are always assumed to be symmetrized; e.g. $M_{A_1}{}^{B_1}\cdots M_{A_n}{}^{B_n}=M_{(A_1}{}^{(B_1}\cdots M_{A_n)}{}^{B_n)}=M_{(A_1}{}^{(B_1}\star\cdots\star M_{A_n)}{}^{B_n)}$, or equivalently, $M_{A_1B_1}\cdots M_{A_nB_n}=M_{\langle A_1B_1}\cdots M_{A_nB_n\rangle}=M_{\langle A_1B_1}\star\cdots\star M_{A_nB_n\rangle}$.

[10] The traceless type-(n,n) projection implies that $\mathbf{P}(M_{AB}M_{C(n),D(n)})=\Delta_n\mathbf{P}M_{[A|\{C(n)|,|B]|D(n)\}}=\Delta_nM_{AC(n),BD(n)}$ with $\mathbf{P}\equiv\mathbf{P}_{\langle AC(n),BD(n)\rangle}$, and (suppressing the anti-symmetry on AB)

$$\begin{aligned} \eta_{\{C_1|[A}\eta_{B]|D_1}M_{C(n-1),D(n-1)\}} &= \eta_{AC_1}\eta_{BD_1}M_{C(n-1),D(n-1)} \\ &\quad+\beta_n\left(\eta_{C_1C_2}\eta_{BD_1}M_{AC(n-2),D(n-1)}+\eta_{AC_1}\eta_{D_1D_2}M_{C(n-1),BD(n-2)}\right. \\ &\quad\left.+\eta_{C_1D_1}\eta_{C_2A}M_{BC(n-2),D(n-1)}+\eta_{C_1D_1}\eta_{D_2B}M_{C(n-1),AD(n-2)}\right) \\ &\quad+\alpha_n\left(\eta_{C_1C_2}\eta_{D_1D_2}M_{AC(n-2),BD(n-2)}-\eta_{C_1D_1}\eta_{C_2D_2}M_{AC(n-2),BD(n-2)}\right)\,, \end{aligned} \tag{2.13}$$

with $\alpha_n=\frac{1}{4}\frac{(n-1)^2}{(n+\epsilon_0-1)(n+\epsilon_0-\frac{1}{2})}$ and $\beta_n=-\frac{1}{2}\frac{n-1}{n+\epsilon_0-1}$.

(ii) the closure relation $[\mathrm{ac}_{M_{AB}}, \mathrm{ac}_{M_{CD}}] M_{E(n),F(n)} \equiv 2i\eta_{BC} \mathrm{ad}_{M_{AD}} M_{E(n),F(n)} - (A \leftrightarrow B)$.

In (i) it suffices to substitute (2.9) to order $\mathcal{O}(\eta^2)$ and $\mathcal{O}(\eta)$ on the left-hand and right-hand sides, respectively, after which contraction by η^{BD_1} and usage of $V \approx 0$ yields λ_n (for fixed n). In (ii) the closure requirement yields an inhomogeneous first-order recursion relation for λ_n with initial datum $\lambda_0 = 0$. The latter method relies on the $\mathfrak{g}$-covariance of the entire procedure of factoring out $\mathcal{I}[V]$ and does therefore not require any explicit usage of $V \approx 0$.

The separate left and right actions of $\mathcal{A}$ on itself induce a left $\mathcal{A}$-module $\mathcal{S}$ and a dual right $\mathcal{A}$-module $\mathcal{S}^*$ characterized by $V \star \mathcal{S} = \mathcal{S}^* \star V = 0$ and with pairing $X(Y) = (X, Y)_{\mathcal{A}} = \mathrm{Tr}[X \star Y]$ where the trace is defined in (2.42). We note that since $X \star X$ contains a unit component for all $X \in \mathcal{A}$ it follows that $\mathcal{S}_X \equiv \mathcal{A} \star X = \mathcal{S}_1 = \mathcal{S}$ and $\mathcal{S}^*_X \equiv X \star \mathcal{A} = \mathcal{S}^*_1 = \mathcal{S}^*$. Thus, as a two-sided module

$$\mathcal{A} \simeq (\mathcal{S} \otimes \mathcal{S}^\star)/\sim\,, \qquad (X \star X') \otimes X'' \;\sim\; X \otimes (X' \star X'')\,, \tag{2.15}$$

where the isomorphism follows from $\mathcal{A} \otimes \mathcal{A} \sim \mathbb{1} \otimes (\mathcal{A} \star \mathcal{A}) = \mathbb{1} \otimes \mathcal{A} \simeq \mathcal{A}$. The values[11] $C_{2n}[\mathcal{S}]$ of the Casimir operators $C_{2n}[\mathfrak{g}] = \frac{1}{2} M_{A_1}{}^{A_2} \star M_{A_2}{}^{A_3} \star \cdots \star M_{A_{2n}}{}^{A_1}$ can be expressed in terms of $C_2[\mathcal{S}]$ using $V_{AB} \approx 0$ which implies $M_A{}^B \star M_{BC} \approx \frac{i(D-1)}{2} M_{AC} + \mu^2 \eta_{AC}$ with $\mu^2 \equiv -\frac{2C_2[\mathcal{S}]}{D+1}$. For example $C_4[\mathcal{S}] = \frac{2}{D+1} C_2[\mathcal{S}]^2 + \frac{(D-1)^2}{4} C_2[\mathcal{S}]$. Using also $V_{ABCD} = M_{[AB} \star M_{CD]} = M_{[AB} \star M_{C]D} - i\eta_{D[A} M_{BC]}$ one finds $M_A{}^B \star V_{BCDE} \approx \left(\mu^2 - \epsilon_0\right) \star \eta_{A[C} M_{DE]}$, from which one deduces that

$$C_2[\mathcal{S}] = -\epsilon_0(\epsilon_0 + 2)\,, \quad C_4[\mathcal{S}] \;=\; (\epsilon_0^2 + \epsilon_0 + 1) C_2[\mathcal{S}]\,, \qquad \epsilon_0 \;\equiv\; \frac{D-3}{2}\,. \tag{2.16}$$

These values equal those of the scalar singleton $\mathfrak{D}_0 \equiv \mathfrak{D}(\epsilon_0; (0))$, as can be seen by using (C.1), and one can show more generally that $C_{2n}[\mathcal{S}] = C_{2n}[\epsilon_0; (0)]$ for all n. One can also show that $\mathcal{I}[V]$ is isomorphic to the scalar-singleton annihilator[12]

$$\mathcal{I}[V] \simeq \mathcal{I}[\mathfrak{D}_0]\,. \tag{2.17}$$

In $D = 4$, also the spinor singleton $\mathfrak{D}_{\frac{1}{2}} \equiv \mathfrak{D}(1; (\frac{1}{2}))$ is annihilated by $\mathcal{I}[V]$, that is

$$D = 4 : \mathcal{I}[V] \;\simeq\; \mathcal{I}[\mathfrak{D}_0] \;\simeq\; \mathcal{I}[\mathfrak{D}_{\frac{1}{2}}]\,. \tag{2.18}$$

2.2 Adjoint and twisted-adjoint $\mathfrak{so}(D+1;\mathbb{C})$ modules

The space $\mathcal{A}$ also provides reducible modules $\mathcal{T}^{(\mathfrak{m}_l)}$ for the generalized adjoint $\mathfrak{g}$ actions

$$\mathrm{ad}^{(\mathfrak{m}_l)}_{M_{AB}} X \;=\; M_{AB} \star X - X \star \pi_{(\mathfrak{m}_l)}(M_{AB})\,, \tag{2.19}$$

[11] If $C_{2n}[\mathfrak{g}]$ assumes a fixed value in a representation $\mathfrak{R}$ we shall denote it by $C_{2n}[\mathfrak{R}] = C_{2n}[\mathfrak{g}|\mathfrak{R}]$.

[12] The elements of an associative algebra $\mathcal{A}$ that act trivially in an $\mathcal{A}$-module $\mathfrak{R}$ generate an ideal $\mathcal{I}[\mathfrak{R}]$ referred to as the $\mathfrak{R}$ annihilator. Eqs. (2.17) and (2.18) follow the lemma that if $X \in \mathcal{U}'[\mathfrak{g}]$ belongs to an adjoint $\mathfrak{g}$ irrep then $X \in \mathcal{I}[\mathfrak{D}]$ where $\mathfrak{D}$ is a lowest-weight representation iff X annihilates the lowest-weight state. If D is even then $V_{AB}|\epsilon_0; \vec{s}_0\rangle = 0$ admits only $|\epsilon_0, (0)\rangle$ and $|\epsilon_0 + \frac{1}{2}, (\frac{1}{2})\rangle$. The condition $V_{ABCD} \approx 0$ rules out the spinor for $D > 4$. Indeed, $C_2[\mathfrak{g}|\epsilon_0; (0)] = -\epsilon_0(\epsilon_0+2)$ and $C_2[\mathfrak{g}|\epsilon_0+\frac{1}{2}; (\frac{1}{2})] = -\frac{1}{2}\left(\epsilon_0 + \frac{1}{2}\right)(\epsilon_0+2)$ are equal iff $D = 4$.

where $\pi_{(\mathfrak{m}_l)}$ are the involutive $\mathcal{A}$-automorphisms $\pi_{(\mathfrak{m}_l)}(M_{AB}) = 2\left(\mathbf{P}_{(\mathfrak{m}_l)}\right)_{AB}{}^{CD} M_{CD} - M_{AB}$ with $\mathbf{P}_{(\mathfrak{m}_l)}$ being the projector onto the generalized Lorentz subalgebras $\mathfrak{m}_l \simeq \mathfrak{so}(D+1-l;\mathbb{C}) \oplus \mathfrak{so}(l;\mathbb{C})$ of $\mathfrak{g}$ for $l = 0, 1, 2, \dots$. Thus, $\mathfrak{g} = \mathfrak{m}_l \oplus\!\!\!\!+ \,\mathfrak{p}_l$ with $[\mathfrak{m}_l, \mathfrak{m}_l]_\star = \mathfrak{m}_l$, $[\mathfrak{m}_l, \mathfrak{p}_l]_\star = \mathfrak{p}_l$ and $[\mathfrak{p}_l, \mathfrak{p}_l] = \mathfrak{m}_l$, and $\pi_{(\mathfrak{m}_l)}(X) = X$ for $X \in \mathfrak{m}_l$ and $\pi_{(\mathfrak{m}_l)}(X) = -X$ for $X \in \mathfrak{p}_l$. The modules $\mathcal{T}^{(\mathfrak{m}_l)}$ decompose into $\mathfrak{g}$ irreps $\mathcal{T}_\ell^{(\mathfrak{m}_l)}$ that we shall refer to as the levels of $\mathcal{T}^{(\mathfrak{m}_l)}$. If $\mathfrak{m}_1 \simeq \mathfrak{m}$ we write

$$\mathcal{L} = \mathcal{T}^{(\mathfrak{g})}\,, \qquad \mathcal{T} \;=\; \mathcal{T}^{(\mathfrak{m})}\,, \tag{2.20}$$

and define the twisted-adjoint action $\widetilde{\mathrm{ad}} \equiv \mathrm{ad}^{(\mathfrak{m})}$ by

$$\widetilde{\mathrm{ad}}_{M_{AB}} X = \widetilde{M}_{AB} X \;=\; M_{AB} \star X - X \star \pi(M_{AB})\,, \qquad \pi \;=\; \pi_{(\mathfrak{m})} \tag{2.21}$$

$$\pi(M_{ab}) = M_{ab}\,, \quad \pi(P_a) \;=\; -P_a\,, \qquad P_a \;=\; M_{\sharp a}\,. \tag{2.22}$$

The automorphism π, which is outer in $\mathcal{A}$, becomes inner[13] in the enlarged algebra[14]

$$\mathcal{A}_k \;=\; \mathcal{A} \oplus (\mathcal{A} \star k)\,, \tag{2.23}$$

where by definition $k \star X = \pi(X) \star k$, $k \star k = \mathbb{1}$, $\tau(k) = \pi(k) = k$ and k acts as π on lowest-weight spaces. As shown in appendix B, the ideal $\mathcal{I}[V]$ has the Lorentz-covariant presentation

$$P^a \star P_a \approx \sigma\epsilon_0\,, \qquad P_{[a} \star P_b \star P_{c]} \;\approx\; 0\,, \tag{2.24}$$

and we note the auxiliary trace constraints $P^a \star M_{ab} \approx M_{ba} \star P^a \approx i(\epsilon_0+1)P_b$ and $M_{(a}{}^c \star M_{b)c} \approx \sigma P_{(a} \star P_{b)} - \epsilon_0 \eta_{ab}$, and that the value of $C_2[\mathfrak{m}] = \frac{1}{2} M_{ab} \star M^{ab}$ in $\mathcal{S}$ (left action) is given by $C_2[\mathfrak{m}|\mathcal{S}] = -\epsilon_0(\epsilon_0+1)$. Thus the elements of $\mathcal{A}$ have the Lorentz-covariant expansions

$$X \approx \sum_{n \geqslant m \geqslant 0} X^{a(n),b(m)} T_{a(n),b(m)}\,, \tag{2.25}$$

where the traceless type-(n,m) basis elements[15]

$$\begin{aligned} T_{a(n),b(m)} \;\equiv\; M_{\{a(n),b(m)\}\sharp(n-m)} \;&=\; M_{\{a_1 b_1} \cdots M_{a_m b_m} P_{a_{m+1}} \cdots P_{a_n\}} \\ &= \sum_{k=0}^{[m/2]} \kappa_{n,m;k} M_{\langle a(n), b(m-2k)\sharp(n-m+2k)} \eta_{b_1 b_2} \cdots \eta_{b_{2k-1} b_{2k}\rangle}\,, \end{aligned} \tag{2.26}$$

[13] The inner automorphisms of a unital associative algebra $\mathcal{A}$ are the adjoint maps Ad_g defined for arbitrary invertible elements $g \in \mathcal{A}$. We note that the latter elements form a group inside $\mathcal{A}$, that one may view as the non-minimal higher-spin group in the case that $\mathcal{A}$ is defined as in (2.8).

[14] The operator k intertwines the extended adjoint and twisted-adjoint modules $\mathfrak{ho}_0 \oplus (\mathcal{T}_+ \star k)$ and $\mathcal{T}_+ \oplus (\mathfrak{ho}_0 \star k)$ defined in (2.52) (although this does not imply the agreement (2.30) and (2.31) between the values of the Casimir operators on the irreducible subspaces). We recall that a linear map $f: V \to W$ is an intertwiner between two $\mathfrak{g}$ modules V and W if f commutes with $\mathfrak{g}$ action, that is $Xf(v) = f(X(v))$ for all $v \in V$ and $x \in \mathfrak{g}$. If f is a vector space isomorphism, then V and W are isomorphic as modules.

[15] The non-monomial basis elements in (2.9) and (2.26) are strongly equal, i.e. as elements in $\mathcal{U}[\mathfrak{g}]$, and obey $\tau(M_{A(n),B(n)}) = (-1)^n M_{A(n),B(n)}$ and $\pi(T_{a(n),b(m)}) = (-1)^{n-m} T_{a(n),b(m)}$

where $M_{A(n),B(n)}$ are given by (2.9) and $\kappa_{n,m;k}$ are fixed by the requirement that $T_{a(n),b(m)}$ is traceless. The resulting *Lorentz-covariant adjoint and twisted-adjoint modules* read

$$\mathcal{L}|_{\mathfrak{g}} = \bigoplus_{\ell\in\{-\frac{1}{2},0,\frac{1}{2},1,\dots\}}^{\infty} \mathcal{L}_\ell\,, \qquad \mathcal{T}|_{\mathfrak{g}} = \bigoplus_{\ell\in\{-1,\frac{1}{2},0,\frac{1}{2},\dots\}}^{\infty} \mathcal{T}_\ell\,, \tag{2.27}$$

respectively, where the ℓth levels take the form

$$Q_\ell|_{\mathfrak{m}} = Q^{A(2\ell+1),B(2\ell+1)} M_{A(2\ell+1),B(2\ell+1)}\,, \quad \ell = -\frac{1}{2}, 0, \frac{1}{2}, 1, \dots\,, \tag{2.28}$$

$$S_\ell|_{\mathfrak{m}} = \sum_{k=0}^{\infty} \frac{i^k}{k!} S^{a(s+k),b(s)} T_{a(s+k),b(s)}\,, \qquad \ell = -1, -\frac{1}{2}, 0, \frac{1}{2}, \dots\,, \quad s = 2\ell+2\,. \tag{2.29}$$

As shown in appendix C, the adjoint and twisted-adjoint levels with $\ell \geqslant -\frac{1}{2}$ have equal quadratic and quartic Casimirs, namely

$$C_2[\mathfrak{g}|\mathcal{L}_\ell] = C_2[\mathfrak{g}|\mathcal{T}_\ell] \;=\; C_2[\ell] \;=\; 2(s-1)(s+2\epsilon_0)\,, \tag{2.30}$$

$$C_4[\mathfrak{g}|\mathcal{L}_\ell] = C_4[\mathfrak{g}|\mathcal{T}_\ell] \;=\; C_4[\ell] \;=\; (s^2+(2\epsilon_0-1)s+2\epsilon_0^2-\epsilon_0+1)C_2[\ell]\,. \tag{2.31}$$

Moreover, as can be seen using (C.1) and (C.2), these values coincide with those of the composite-massless lowest-weight spaces $\mathfrak{D}(s+2\epsilon_0;(s))$, i.e.

$$C_2[\ell] = C_2[\mathfrak{g}|s+2\epsilon_0;(s)]\,, \qquad C_4[\ell] \;=\; C_4[\mathfrak{g}|s+2\epsilon_0;(s)]\,. \tag{2.32}$$

As we shall see, these agreements follow from direct relationships between $\mathcal{L}_\ell$, $\mathcal{T}_\ell$ and $\mathfrak{D}(\pm(s+2\epsilon_0);(s))$ visible in the corresponding $\mathfrak{so}(2)\oplus\mathfrak{so}(D-1)$-covariant weight-space modules. In $\mathcal{T}_\ell$, which is infinite-dimensional, the required change of basis is non-trivial, and actually amounts to the *harmonic expansion* of the linearized Weyl tensors. One therefore distinguishes between the Lorentz-covariant slicing (2.27), where the twisted-adjoint elements are arbitrary polynomials in $\mathcal{A}$, and the corresponding *twisted-adjoint compact-weight modules*

$$\mathcal{M}(\sigma,\sigma')|_{\mathfrak{g}} = \bigoplus_{s} \mathcal{M}_{(s)}(\sigma,\sigma')\,, \qquad \mathcal{M}_{(s)}(\sigma,\sigma')|_{\mathfrak{h}} \;=\; \bigoplus_{\kappa} \mathbb{C}\otimes T^{(s)}_\kappa\,, \tag{2.33}$$

where the elements by definition are arbitrary polynomials in compact basis elements $T^{(s)}_\kappa$ that belong to finite-dimensional (unitary) representations of $\mathfrak{h}$. The latter are given by series expansions in $T_{a(s+k),b(s)}$ $(k=0,1,\dots)$ except for $\mathcal{M}_{(s)}(+,-)$, in which case $\mathfrak{m}=\mathfrak{h}$.

2.3 Adjoining the adjoint and composite-massless representations

We propose that the Casimir relations (2.30), (2.31) and (2.32) follow from that $\mathcal{L}_\ell$, $\mathcal{M}_\ell(+,+)$ and $\mathfrak{D}(s+2\epsilon_0;(s))$, where $s = 2\ell+2$, can actually be adjoined in the $\mathfrak{so}(2)\oplus\mathfrak{so}(D-1)$ weight space. As shown in section 3.1, the space $\mathfrak{D}(s+2\epsilon_0;(s))$ arises upon harmonic expansion inside $\mathcal{M}(+,+)$. Moreover, we propose that $\mathfrak{D}(s+2\epsilon_0;(s))$ is adjoined to the lowest-and-highest-weight space $\mathfrak{D}(-s+1;(s-1)) \simeq \mathcal{L}_\ell$ via the intermediate conjugate-massless[16] lowest-weight space $\mathfrak{D}(-s+2;(s))$ (see figure 1) in such a way that the Harish-Chandra modules

$$0 \hookrightarrow \mathfrak{C}(-s+1;(s-1)) \longrightarrow \mathfrak{C}(-s+2;(s)) \longrightarrow \mathfrak{C}(s+2\epsilon_0;(s)) \longrightarrow 0\,, \tag{2.34}$$

enter a short exact sequence for certain Young-projected actions of L^+_r. First, the ground state of $\mathfrak{C}(-s+2;(s))$ has the same quantum numbers as the singular vector

$$|-s+2;(s)\rangle = L^+_{\{r_1}|-s+1;(s-1)\rangle_{r(s-1)\}} \in \mathfrak{I}(-s+1;(s-1))\,. \tag{2.35}$$

Next, we propose that the lowest-weight state of $\mathfrak{C}(s+2\epsilon_0;(s))$ has the same quantum numbers as the following singular vectors in $\mathfrak{I}(-s+2;(s))$:

$$D=4: \qquad |s+1;(s)\rangle_{r(s)} = \epsilon_{r_1t_1u_1}\cdots\epsilon_{r_st_su_s}L^+_{u_1}\cdots L^+_{u_{s-1}}|2;(s,1)\rangle_{t(s),u_s}\,, \tag{2.36}$$

$$D\geqslant 5: \qquad |s+2\epsilon_0;(s)\rangle_{r(s)} = (x^+)^{\frac{D-5}{2}}L^+_{t_1}\cdots L^+_{t_s}|2;(s,s)\rangle_{r(s),t(s)}\,, \tag{2.37}$$

where $x^+\equiv L^+_rL^+_r$ and $|2;(s,s)\rangle$ and $|2;(s,1)\rangle_{t(s),u_s}$ are the descendants of $|-s+2;(s)\rangle$ given by

$$D\geqslant 5: \qquad |2;(s,s)\rangle_{r(s),t(s)} = L^+_{\{r_1}\cdots L^+_{r_s}|-s+2;(s)\rangle_{t(s)\}}\,, \tag{2.38}$$

$$D=4: \qquad |2;(s,1)\rangle_{r(s),t} = \mathbf{P}_{\{s,1\}}\left(\prod_{i=1}^{s-1}\epsilon_{r_iu_iv_i}L^+_{u_i}\right)|-s+2;(s)\rangle_{v(s-1)t}\,, \tag{2.39}$$

where $\mathbf{P}_{\{s,1\}}$ denote the traceless type-$(s,1)$ projection of the index group $r(s),t$. We have checked that (2.36) and (2.37) are singular vectors for $s=1$ and $s=2$ in $D=4$ and $D=5$. In $D=6,8,\dots$ we define $\mathfrak{g}$ action on $\sqrt{x^+}$ by extending

$$L^-_r(x^+)^n = (x^+)^nL^-_r + 4n(x^+)^{n-1}\left(iL^+_sM_{rs}+L^+_r(E+n-\epsilon_0-1)\right)\,, \tag{2.40}$$

which is valid in $\mathcal{U}[\mathfrak{g}]$, to arbitrary differentiable functions $f(x^+)$ as follows:

$$[L^-_r, f(x^+)] = 4\frac{d}{dx^+}f(x^+)\left(iL^+_sM_{rs}+L^+_r(E-\epsilon_0)\right) + 4x^+\frac{d^2}{d(x^+)^2}f(x^+)L^+_r\,. \tag{2.41}$$

The singular vectors (2.37) and (2.36) vanish formally if $|-s+2;(s)\rangle$ is substituted with (2.35), so that $|-s+2;(s)\rangle$ and $|2;(s,s)\rangle$) or $|2;(s,1)\rangle$ are weight-space analogs of an abelian gauge field and its Weyl or Cotton tensor, respectively. In other words, the sequence (2.34) is the weight-space counterpart of the Chevalley-Eilenberg cocycle appearing in the linearized one-form constraint (2.68).

[16]The composite-massless and conjugate-massless lowest-weight states are related by the "conjugation" $e_0 \to D-1-e_0$, that leaves invariant the values of the Casimir operators (see eqs. (C.1) and (C.2)). Following a standard terminology (see, for example, [38] and references therein), we will sometimes refer to them as *shadow* fields.

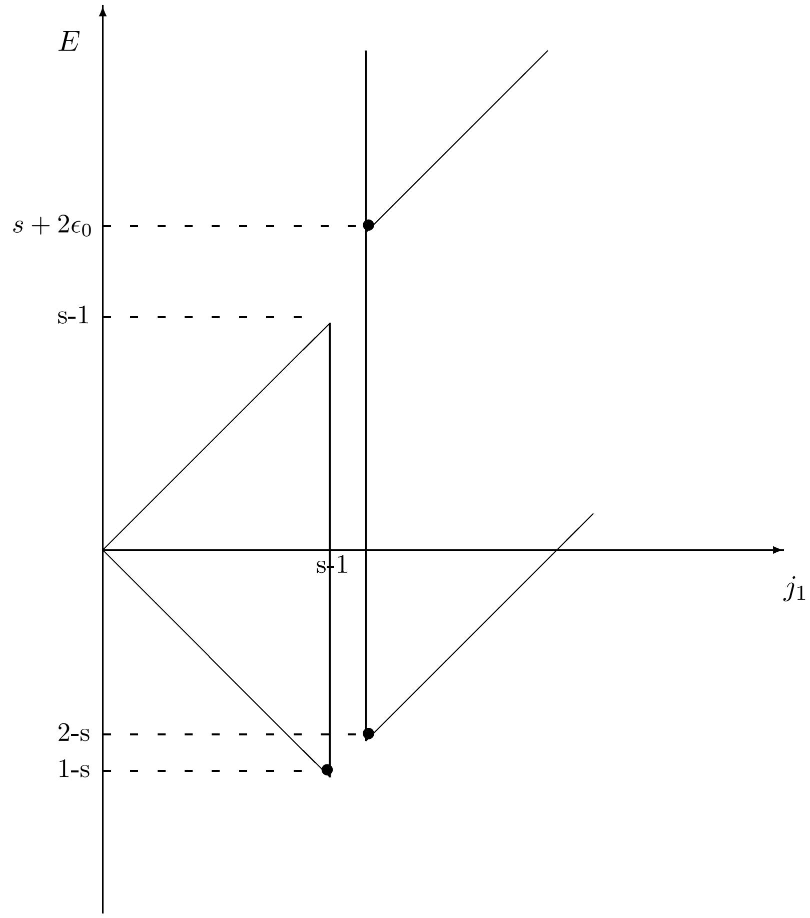

Figure 1: The adjoint module $\mathfrak{D}(-(s-1);(s-1))$ is connected via the conjugate massless module $\mathfrak{D}(-(s-2);(s))$ to the massless module $\mathfrak{D}(s+2\epsilon_0;(s))$.

2.4 Non-composite trace and reflectors

A trace on an associative algebra $\mathcal{A}$ is a linear map $\mathrm{Tr}:\mathcal{A}\to\mathbb{C}$ that obeys $\mathrm{Tr}(X\star Y-Y\star X)=0$. If $\mathcal{A}$ contains a Lie subalgebra $\mathfrak{l}$ and decomposes under $\mathrm{ad}_{\mathfrak{l}}$ into finite-dimensional irreps $\mathcal{A}_\lambda$, that is $\mathcal{A}|_{\mathfrak{l}}=\bigoplus_\lambda \mathcal{A}_\lambda$, $\mathcal{A}\ni X=\sum_\lambda T_\lambda X_\lambda$, and if the non-singlets $T_{\lambda'}$ are expressible as $\star$-commutators, then $\mathrm{Tr}[X]=\mathrm{Tr}_{\mathcal{A}|\mathfrak{l}}[X]=\sum_{\lambda_0\in\Lambda}t_{\lambda_0}X_{\lambda_0}$, provided the trace is finite, where λ_0 label the $\mathfrak{l}$ singlets and t_{λ_0} are complex numbers compatible with cyclicity. If $\mathcal{A}[\mathfrak{g}|\mathfrak{R}]\equiv\mathcal{U}[\mathfrak{g}]/\mathcal{I}[\mathfrak{R}]$ where $\mathfrak{g}$ is a finite-dimensional Lie algebra with module $\mathfrak{R}$, then the *non-composite trace*[17] is defined as $\mathrm{Tr}_{\mathcal{A}[\mathfrak{g}|\mathfrak{R}]|\mathfrak{g}}[X]\equiv X_{\mathbf{1}}$ where $X_{\mathbf{1}}$ is the coefficient of $\mathbb{1}\in\mathcal{A}[\mathfrak{g}|\mathfrak{R}]$ in a *preferred basis*. This trace is well-defined since $X\approx X'$ iff X and X' have the same expansions in the preferred basis. The cyclicity follows from $\mathrm{Tr}[\tau(X)]=\mathrm{Tr}[X]$ where τ is the anti-automorphism defined in (2.2), which implies that if $X,Y\in\mathcal{A}[\mathfrak{g}|\mathfrak{R}]$ then $\mathrm{Tr}[X\star Y]=\mathrm{Tr}[\tau(X\star Y)]=\mathrm{Tr}[\tau(Y)\star\tau(X))]=\mathrm{Tr}[Y\star X]$, as can be seen by splitting $X=X_++X_-$ with $\tau(X_\pm)=\pm X_\pm$, idem Y, and noting that $\mathrm{Tr}(X_\pm\star Y_\mp)=0$. If $\mathcal{A}[\mathfrak{g}|\mathfrak{R}]$

[17] The non-composite trace of the universal enveloping algebra $\mathcal{U}[\mathfrak{g}]$ is trivial in the sense that $\mathrm{Tr}[X\star Y]=(X\star Y)_{\mathbf{1}}=X_{\mathbf{1}}Y_{\mathbf{1}}$.

is infinite-dimensional, which requires $\mathfrak{R}$ to be infinite-dimensional, the non-composite and composite traces[18] are in general not equivalent.

In the case of $\mathcal{A} = \mathcal{A}[\mathfrak{so}(D+1;\mathbb{C})|\mathfrak{D}_0]$ we therefore distinguish between the composite trace $\mathrm{Tr}_{\mathfrak{D}_0}$ over the scalar singleton (see appendix A.2) and the non-composite trace, and define

$$\mathrm{Tr}[X] = X_0 \ = \ X^{(0,0)} \, , \tag{2.42}$$

where X_0 is the $\mathfrak{g}$-singlet in (2.8) defined with respect to the preferred basis elements (2.9), and $X^{(0,0)}$ is the corresponding Lorentz singlet in (2.25) defined with respect to the preferred basis elements (2.26). These singlets are equal since the preferred basis elements are equal (strongly) in $\mathcal{U}[\mathfrak{g}]$, and the (n,n)-plet of $\mathfrak{g}$ contains the singlet of $\mathfrak{m}$ iff $n = 0$. The trace Tr extends uniquely to $\mathcal{A}_k$ defined in (2.23) since $\mathcal{A} \star k$ does not contain any singlet, which implies $\mathrm{Tr}[X \star k] = 0$. As shown in appendix G.2, the Tr is equivalent in $D = 4$ to the composite supertrace

$$D = 4 \quad \mathrm{Tr} \ = \ \mathrm{Tr}_{\mathfrak{D}_0} - \mathrm{Tr}_{\mathfrak{D}_{1/2}} \, . \tag{2.43}$$

The trace Tr can equivalently be written as

$$\mathrm{Tr}[X] = {}_{12}\langle \mathbb{1}^* | X(1) | \mathbb{1} \rangle_{12} \, , \tag{2.44}$$

where the *non-composite reflector* $|\mathbb{1}\rangle_{12} \in \mathcal{B}$ and its dual ${}_{12}\langle \mathbb{1}^*| \in \mathcal{B}^*$ are elements in the left and right $\mathcal{A}_k$-bimodules[19]

$$\mathcal{B} = \{ |X\rangle_{12} : \ V(\xi)|X\rangle_{12} = 0 \} \, , \qquad \mathcal{B}^* \ = \ \{ {}_{12}\langle X^*| : \ {}_{12}\langle X^* | V(\xi) = 0 \} \, , \tag{2.45}$$

where $\xi = 1, 2$, obeying the normalization condition ${}_{12}\langle \mathbb{1}^* | \mathbb{1}\rangle_{12} = 1$ and the overlap condition

$$(X(1) - (\tau \circ \pi)(X)(2)) \, |\mathbb{1}\rangle_{12} = 0 \, , \qquad {}_{12}\langle \mathbb{1}^* | \, (X(1) - (\tau \circ \pi)(X)(2)) \ = \ 0 \, , \tag{2.46}$$

for all $X \in \mathcal{A}_k$ and where $\tau \circ \pi$ is the anti-automorphism composed by τ defined in (2.2) and π defined in (2.22). The overlap conditions and $(\pi\tau)^2 = \mathrm{Id}$ imply that ${}_{12}\langle \mathbb{1}^* | (X \star Y)(1) | \mathbb{1}\rangle_{12} = {}_{12}\langle \mathbb{1}^* | (Y \star X)(1) | \mathbb{1}\rangle_{12}$ so that ${}_{12}\langle \mathbb{1}^* | [M_{AB}(1), X(1)]_\star | \mathbb{1}\rangle_{12} = 0$. Since

[18] If $\mathfrak{R}$ has a finite dimension $|\mathfrak{R}|$ then the algebra $\mathcal{A}[\mathfrak{g}|\mathfrak{R}] \subset GL(|\mathfrak{R}|;\mathbb{C})$, and the non-composite trace is equivalent to the composite trace $\mathrm{Tr}_{\mathfrak{R}}[X] = \sum_n \langle n^* | X | n \rangle$ with $|n\rangle$ and $\langle n^*|$ being basis elements for $\mathfrak{R}$ and $\mathfrak{R}^*$ with normalization $\langle m^* | n \rangle = \delta^n_m$. The composite trace can be written as $\mathrm{Tr}_{\mathfrak{R}}[X] = {}_{12}\langle \mathbf{1}^*_{\mathfrak{R}} | X(1) | \mathbf{1}\rangle_{12}$ where $|\mathbf{1}_{\mathfrak{R}}\rangle_{12} = \sum_n |n\rangle_1 \otimes |n\rangle_2$ and ${}_{12}\langle \mathbf{1}^*_{\mathfrak{R}}| = \sum_n {}_1\langle n^*| \otimes {}_2\langle n^*|$ are referred to as composite reflectors of $\mathfrak{R}$.

[19] A left bimodule is a left 2-module, where, in general, a left P-module V, $P \in \mathbb{N}$, is a linear space with the property that if $v \in V$ and $X, Y \in \mathcal{A}$, then $X(\xi)v \in V$ and $(X \star Y)(\xi)v = X(\xi)(Y(\xi)v)$ for $\xi = 1, \ldots, P$, and $X(\xi)Y(\eta)v = Y(\eta)X(\xi)v$ if $\xi \neq \eta$. We recall that if V is an $\mathcal{A}$ left module then V^* is a $\mathcal{A}$ right module via $v^*(Xw) = (v^*X)(w)$ for $v^* \in V^*$, $w \in V$ and $X \in \mathcal{A}$. This generalizes straightforwardly to P-modules by defining $v^*(X(\xi)w) = (v^*X(\xi))(w)$. Finite-dimensional bimodules are composite in the sense that their elements are sums of factorized elements of the form $u_1 \otimes v_2$, while the analog need not hold for infinite-dimensional bimodules. We also note that since $\pi(V) = V$ it follows that if $|X\rangle_{12} \in \mathcal{B}$ and ${}_{12}\langle X^*| \in \mathcal{B}^*$, then $k(\xi)|X\rangle_{12} \in \mathcal{B}$ and ${}_{12}\langle X^*|k(\xi) \in \mathcal{B}^*$ for $\xi = 1, 2$.

JHEP10(2008)022

X_n are $\mathfrak{g}$-commutators for $n>0$ it follows that ${}_{12}\langle \mathbb{1}^*|X(1)|\mathbb{1}\rangle_{12} ={}_{12}\langle \mathbb{1}^*|\mathbb{1}\rangle_{12} X_0 = \mathrm{Tr}[X]$. Moreover, taking the expectation value of $(P^a\star k\star P_a - \sigma\epsilon_0 k)(1) = 0$ implies that ${}_{12}\langle \mathbb{1}^*|k(1)|\mathbb{1}\rangle_{12} = 0$. From (2.46) it follows that $\mathcal{O}_{12}\equiv_{23}\langle \mathbb{1}^*|\star|\mathbb{1}\rangle_{13}$ obeys $X\star\mathcal{O}=\mathcal{O}\star X$ for all $X\in\mathcal{A}_k$ so that $\mathcal{O}=c\mathbb{1}$. Defining

$$ {}_{23}\langle \mathbb{1}^*|\star|\mathbb{1}\rangle_{13} \;=\; \mathbb{1}_{12}\;, \tag{2.47}$$

one has the sequence of *reflection maps* $\mathcal{B}_{12}\xrightarrow{R_2}(\mathcal{A}_k)_{12}\xrightarrow{R_1}\mathcal{B}^*_{12}$ given by

$$ |X\rangle_{12} \;=\; X(1)|\mathbb{1}\rangle_{12}\,, \qquad {}_{12}\langle X^*| \;=\; {}_{12}\langle \mathbb{1}^*|X(1)\;, \tag{2.48}$$

$$ X_{12} \;=\; {}_{23}\langle \mathbb{1}^*|\star|X\rangle_{13} \;=\; {}_{23}\langle X^*|\star|\mathbb{1}\rangle_{13}\;. \tag{2.49}$$

We note that the condition (2.46) implies, in particular, that $|\mathbb{1}\rangle_{12}$ is a $\mathfrak{m}_{\rm diag}$-invariant element.

We also define the twisted non-composite reflectors

$$ |\widetilde{\mathbb{1}}\rangle_{12} \;=\; k(1)|\mathbb{1}\rangle_{12} \;=\; k(2)|\mathbb{1}\rangle_{12}\,, \qquad {}_{12}\langle\widetilde{\mathbb{1}}^*| \;=\; {}_{12}\langle \mathbb{1}^*|k(1) \;=\; {}_{12}\langle \mathbb{1}^*|k(2)\,, \tag{2.50}$$

obeying the overlap conditions $(X(1)-\tau(X)(2))|\widetilde{\mathbb{1}}\rangle_{12}={}_{12}\langle\widetilde{\mathbb{1}}^*|(X(1)-\tau(X)(2))=0$, which in particular imply that the twisted reflectors are $\mathfrak{g}_{\rm diag}$-invariant. The twisted reflectors have the normalizations ${}_{12}\langle\widetilde{\mathbb{1}}^*|\widetilde{\mathbb{1}}\rangle_{12}=1$, ${}_{12}\langle\widetilde{\mathbb{1}}^*|\mathbb{1}\rangle_{12}={}_{12}\langle\mathbb{1}^*|\widetilde{\mathbb{1}}\rangle_{12}=0$ and ${}_{23}\langle\widetilde{\mathbb{1}}^*|\star|\widetilde{\mathbb{1}}\rangle_{12}=\mathbb{1}_{13}$, ${}_{23}\langle\widetilde{\mathbb{1}}^*|\star|\mathbb{1}\rangle_{12}={}_{23}\langle\mathbb{1}^*|\star|\widetilde{\mathbb{1}}\rangle_{12}=k_{13}$, and Tr now admits the manifestly $\mathfrak{g}$-invariant form

$$ \mathrm{Tr}[X] \;=\; {}_{12}\langle\widetilde{\mathbb{1}}^*|X(1)|\widetilde{\mathbb{1}}\rangle_{12}\;. \tag{2.51}$$

2.5 The adjoint and twisted-adjoint higher-spin representations

The adjoint and twisted-adjoint actions of $\mathcal{A}$ on itself induces the minimal bosonic higher-spin algebra $\mathfrak{ho}_0=\mathfrak{ho}_0(D+1;\mathbb{C})$ and its even and odd twisted-adjoint representations:

$$ \mathfrak{ho}_0 \;\equiv\; \{Q\in\mathcal{A}:\tau(Q) \;=\; -Q\}\,, \qquad \mathcal{T}_\pm \;\equiv\; \{S_\pm\in\mathcal{A}:\tau(S_\pm) \;=\; \pm\pi(S_\pm)\}\;, \tag{2.52}$$

where τ and π are given in (2.2) and (2.22), respectively. Their level decompositions read $\mathfrak{ho}_0|_{\mathfrak{g}}=\bigoplus_{\ell=0}^\infty\mathcal{L}_\ell$ and $\mathcal{T}_\pm|_{\mathfrak{g}} \;=\; \bigoplus_{\ell=-1}^\infty\mathcal{T}_{\ell+\frac12(1\mp1)}$ with levels $\mathcal{L}_\ell$ and $\mathcal{T}_\ell$ given in (2.27). The $\mathfrak{ho}_0$ transformations are defined by

$$ \mathrm{ad}(Q)(Q') = [Q,Q']_\star \;=\; Q\star Q' - Q'\star Q\;. \tag{2.53}$$

$$ \widetilde{\mathrm{ad}}_Q(S) = [Q,S]_\pi \;=\; \widetilde{Q}S \;=\; Q\star S - S\star\pi(Q)\,, \tag{2.54}$$

and mix the levels as follows (see [6, 23, 40, 42] and also [43] for a more recent discussion): $\widetilde{\mathrm{ad}}_{Q_\ell}(S_{\ell'}) = \sum_{\ell''=\max(-1,\ell'-\ell)}^{\ell+\ell'} S_{\ell''}$ and $[Q_\ell,Q_{\ell'}]_\star=\sum_{\ell''=|\ell-\ell'|}^{\ell+\ell'}Q_{\ell''}$, where the lower bound on ℓ'' in the twisted-adjoint case follows from the contraction rules $V_{AB}\approx V_{ABCD}\approx 0$. The algebra $\mathfrak{ho}_0$ is a minimal Lie-algebra extension of $\mathfrak{g}$ in the sense that if $\mathcal{L}'$ is a Lie subalgebra obeying $\mathfrak{g}\subseteq\mathcal{L}'\subseteq\mathfrak{ho}_0$ then either $\mathcal{L}'=\mathfrak{g}$ or $\mathcal{L}'=\mathfrak{ho}_0$. The minimal set-up can be extended in various ways by enlarging the underlying associative algebra $\mathcal{A}$, e.g. to $\mathcal{A}_k$ which corresponds to adding extra auxiliary fields (see, for example, [22] and references therein), and/or by tensoring with internal associative algebras which adds Yang-Mills-like sectors, and fermionic generators [5, 6, 33, 44]. The in some sense simplest extension is to add the half-integer levels, which leads to the non-minimal bosonic higher-spin algebra

$$\mathfrak{ho} \equiv \mathcal{L}\,, \qquad \mathfrak{ho}|_{\mathfrak{g}} \;=\; \bigoplus_{\ell=-\frac{1}{2},0,\frac{1}{2},1,\dots} \mathcal{L}_\ell\,, \tag{2.55}$$

acting irreducibly in $\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_-$, such that if $\tau(Q_\pm) = \mp Q_\pm$ and $\tau(S_\pm) = \pm\pi(S_\pm)$ then $\mathrm{ad}_{Q_\pm}(Q_\epsilon) = Q_{\pm\epsilon}$ and $\widetilde{\mathrm{ad}}_{Q_\pm}(S_\epsilon) = S_{\pm\epsilon}$ for $\epsilon = \pm$.

The trace operation Tr induces the $\mathfrak{ho}$-invariant bilinear forms on $\mathfrak{ho}$ and $\mathcal{T}$:

$$(Q,Q')_{\mathfrak{ho}} \;=\; -\mathrm{Tr}[Q\star Q'] \;=\; -{}_{12}\langle Q|Q'\rangle_{12}\,, \tag{2.56}$$

$$(S,S')_{\mathcal{T}} \;=\; \mathrm{Tr}[\pi(S)\star S'] \;=\; {}_{12}\langle \pi(S)|S'\rangle_{12}\,, \tag{2.57}$$

where (2.48) has been used to define $|Q_\pm\rangle_{12} = Q_\pm(1)|\mathbb{1}\rangle_{12} = \frac{1}{2}(Q_\pm(1) \mp \pi(Q_\pm)(2))|\mathbb{1}\rangle_{12}$ and $|S_\pm\rangle_{12} = S_\pm(1)|\mathbb{1}\rangle_{12} = \frac{1}{2}(S_\pm(1) \pm S_\pm(2))|\mathbb{1}\rangle_{12}$, and ${}_{12}\langle Q_\pm| = {}_{12}\langle \mathbb{1}^*|Q_\pm(1) = \frac{1}{2}{}_{12}\langle \mathbb{1}^*|(Q_\pm(1)\mp\pi(Q_\pm)(2))$ and ${}_{12}\langle \pi(S_\pm)| = {}_{12}\langle \mathbb{1}^*|\pi(S_\pm)(1) = \frac{1}{2}{}_{12}\langle \mathbb{1}^*|(\pi(S_\pm)(1)\pm\pi(S_\pm)(2))$. These states carry the higher-spin representations

$$|\mathrm{ad}_Q Q'\rangle_{12} \;=\; \widetilde{Q}|Q'\rangle_{12}\,, \qquad |\widetilde{\mathrm{ad}}_Q S\rangle_{12} \;=\; Q|S\rangle_{12}\,, \tag{2.58}$$

and ${}_{12}\langle \mathrm{ad}_Q Q'| = -{}_{12}\langle Q'|\widetilde{Q}$, ${}_{12}\langle \pi(\widetilde{\mathrm{ad}}_Q S)| = -{}_{12}\langle \pi(S)|Q$, where

$$\widetilde{Q}|X\rangle_{12} \equiv (Q(1) + \pi(Q)(2))|X'\rangle_{12}\,, \qquad Q|S\rangle_{12} \;\equiv\; (Q(1) + Q(2))|S\rangle_{12}\,. \tag{2.59}$$

The $\mathfrak{ho}$ action is (anti-)self-adjoint with respect to the inner products (2.56) and (2.57), that is

$$(\mathrm{ad}_Q Q', Q'')_{\mathfrak{ho}} = -(Q', \mathrm{ad}_Q Q'')_{\mathfrak{ho}}\,, \qquad (\widetilde{\mathrm{ad}}_Q S, S')_{\mathcal{T}} \;=\; -(S, \widetilde{\mathrm{ad}}_Q S')_{\mathcal{T}}\,. \tag{2.60}$$

The real forms of $\mathfrak{ho}$ and $\mathcal{T}$ are given by

$$Q^\dagger \;=\; -Q\,, \qquad S^\dagger \;=\; \pi(S)\,, \tag{2.61}$$

where $\dagger$ acts as $(M_{AB})^\dagger = M_{AB}$ and as standard complex conjugation of coefficients.

2.6 Adjoint and twisted-adjoint master fields

The unfolded formulation of a field theory (see, for example, [22, 23] and references therein) involves a space $\mathcal{R} = \bigoplus_\alpha \Omega^{[p_\alpha]} \otimes \mathcal{R}^\alpha$ of locally defined rank-p_α differential forms X^α taking their values in $\mathfrak{g}$ modules $\mathcal{R}^\alpha$ and obeying generalized curvature constraints $R^\alpha \equiv dX^\alpha + f^\alpha(X^\beta) = 0$, where the functions f^α (that are sums of multi-linear products) obey generalized Jacobi identities $f^\beta\partial_\beta f^\alpha \equiv 0$ assuring gauge and diffeomorphism invariance (for arbitrary base manifolds). The corresponding operator $Q = f^\alpha\partial_\alpha$ is nilpotent and, upon expansion around backgrounds, its linearization induces a (locally defined) cohomology in the space $\mathcal{R}^{[*]} = \bigoplus_\alpha \Omega^{[*]} \otimes \mathcal{R}^\alpha$ containing the standard gauge parameters, gauge fields, and field equations, while the non-closed and trivial elements encode Stückelberg symmetries, auxiliary fields and Bianchi identities [23, 25].

In Vasiliev's frame-like formulation of higher-spin gauge theory, the basic building blocks are master fields in $\Omega^{[0]}$ and $\Omega^{[1]}$. The *full* master fields are valued in representations of an extended higher-spin algebra $\widehat{\mathcal{A}}$, and obey full constraints of a remarkably simple form [20, 21, 39, 40]. The full fields can be expanded perturbatively in terms of *reduced* master fields, namely a one-form A and a zero-form Φ, referred to as the Weyl zero-form. In the minimal bosonic model the reduced fields are

$$A \in \Omega^{[1]} \otimes \mathfrak{ho}_0 \,, \qquad \Phi \;\in\; \Omega^{[0]} \otimes \mathcal{T}_+ \,, \tag{2.62}$$

where $\mathfrak{ho}_0$ and $\mathcal{T}_+$ are defined in (2.52). More explicitly,

$$A=\sum_{s=2,4,6,\dots} A_{(s)} \,, \quad A_{(s)}=-i\sum_{t=0}^{s-1} dx^M A_{M,a(s-1),b(t)}(x^N) M^{a_1b_1}\cdots M^{a_tb_t}P^{a_{t+1}}\cdots P^{a_{s-1}} \,, \tag{2.63}$$

$$\Phi=\sum_{s=0,2,4,\dots} \Phi_{(s)} \,, \quad \Phi_{(s)}=\sum_{k=0}^{\infty}\frac{i^k}{k!}\Phi^{a(s+k),b(s)}(x^M) M_{a_1b_1}\cdots M_{a_sb_s}P_{a_{s+1}}\cdots P_{a_{s+k}} \,, \tag{2.64}$$

where x^M are the coordinates of the base manifold. The perturbative constraints have the form

$$F+\sum_{n=1}^{\infty} J_{(n)}(A,A;\Phi,\dots,\Phi) = 0 \,, \qquad D\Phi+\sum_{n=2}^{\infty} P_{(n)}(A;\Phi,\dots,\Phi) \;=\; 0 \,, \tag{2.65}$$

where $F \equiv dA + A\star A$ and $D\Phi \equiv d\Phi + [A,\Phi]_\pi$, and $J_{(n)} : (\mathfrak{ho}_0)^{\otimes 2} \otimes (\mathcal{T}_+)^{\otimes n} \to \mathfrak{ho}_0$ and $P_{(n)} : \mathfrak{ho}_0 \otimes (\mathcal{T}_+)^{\otimes n} \to \mathcal{T}_+$ are multi-linear products that can be computed using the oscillator realizations (see [39] for discussions). The standard formulation in D-dimensional spacetime follows upon splitting $A = A_{(2)} + W + K$ where

$$A_{(2)} \;=\; -i\left(e^a P_a + \frac{1}{2}\omega^{ab}M_{ab}\right) \;=\; -idx^\mu\left(e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab}M_{ab}\right) , \tag{2.66}$$

contains the microscopic Vasiliev-frame vielbein $e_\mu{}^a$ and Lorentz connection $\omega_\mu{}^{ab}$; $W = \sum_{s=4,6,\dots} A_{(s)}$; and K is a field redefinition such that $e_\mu{}^a$ and the component fields in W and Φ transform as tensors under the canonical Lorentz transformations [22, 39, 41]. If $A_{(2)}$ is treated exactly while W and Φ are treated as weak fields, then the expansion of (2.65) yields a set of manifestly diffeomorphism and locally Lorentz invariant constraints, where thus ω appears only via $\nabla \equiv d+\omega$ or $R \equiv d\omega+\omega^2$. If $e_\mu{}^a$ is invertible, then exhaustion of the resulting algebraic constraints and Stückelberg symmetries leaves a set of dynamical microscopic fields: a *scalar field*, a *metric* and a tower of *doubly-traceless symmetric rank-s tensor gauge fields* ($s = 4,6,\dots$) given by:[20]

$$\phi = \Phi_{(0)}|_{P_a=0} \,, \quad g_{\mu\nu} \;=\; e_\mu{}^a e_{\nu a} \,, \quad \phi_{a(s)} \;=\; (e^{-1})_{(a}{}^{\mu} W_{\mu,a(s-1))} \,. \tag{2.67}$$

[20] There are exceptions already in the minimal case, such as the chiral models in Euclidean and Kleinian signatures in $D = 4$ [37], whose metric and tensor-gauge fields are half-flat, say with left-handed curvatures, leaving the right-handed components in Φ as independent dynamical fields.

Among the auxiliary zero-forms are the *generalized spin-s Weyl tensors* $\Phi_{a(s+k),b(s)}$ with $s = 0, 2, 4, \dots$ and $k = 0, 1, 2, \dots$, given on-shell for $s \neq 2$ by s curls and k gradients of $\phi_{a(s)}$ plus higher-order corrections in weak fields, and for $s = 2$ by the kth gradient of the full spin-2 Weyl tensor plus weak-field corrections. At first order in weak fields

$$\mathcal{R} + \nabla W - ie^a\{P_a, W\}_\star = -\frac{i}{2} \sum_{s=2,4,6,\dots} e^a \wedge e^b \Phi_{ac(s-1),bd(s-1)} M^{c_1 d_1} \cdots M^{c_{s-1} d_{s-1}} , \quad (2.68)$$

$$\nabla \Phi - ie^a\{P_a, \Phi\}_\star = 0 , \quad (2.69)$$

where $\mathcal{R} = -i(T^a P_a + \frac{1}{2}(R^{ab} + \sigma e^a e^b) M_{ab}$, $T^a = \nabla e^a = de^a + \omega^{ab} e_b$, $R^{ab} = d\omega^{ab} + \omega^{ac}\omega_c{}^b$ and $\nabla = d - \frac{i}{2}\omega^{ab} M_{ab}$. Since $\mathcal{D} \equiv d + A_{(2)} = \nabla - ie^a P_a$ obeys $\mathcal{D}^2 \equiv \mathcal{R}$, which starts as a weak field, it follows that (2.68) and (2.69) are consistent in the first order:[21] the consistency of (2.68) requires $e^a \wedge e^b \wedge e^c \nabla_a \Phi_{bd(s-1),ce(s-1)} = 0$, which follows from (2.69), which is in its turn consistent. Combining (2.64) and (2.69) with (A.26) yields the following component form of the zero-form constraint (see appendix D):

$$\nabla_c \Phi_{a(s+k),b(s)} - 2k\Delta_{s+k-1,s} \eta_{c\{a} \Phi_{a(s+k),b(s)\}} + \frac{2\lambda^{(s)}_{k+1}}{k+1} \Phi_{c\{a(s+k),b(s)\}} = 0 . \quad (2.70)$$

Using $\Phi_{\langle c\langle a(s+k),b(s)\rangle\rangle} = \Phi_{ca(s+k),b(s)}$ one finds that

$$\Phi_{a(s+k),b(s)} = \frac{(-1)^k k!}{2^k \prod_{l=1}^k \lambda^{(s)}_l} \nabla_{\{a_1} \cdots \nabla_{a_k} \Phi_{a(s),b(s)\}} , \qquad k = 1, 2, \dots . \quad (2.71)$$

Other projections of (2.70) yield the following Bianchi identities and mass-shell conditions:

$$\nabla_{[\mu} \Phi_{\nu|a(s+k-1),|\rho]b(s-1)} = 0 , \qquad s \geqslant 1 , \quad (2.72)$$

$$(\nabla^2 - M^2_{s,k}) \Phi_{a(s+k),b(s)} = 0 , \qquad s \geqslant 0 , \quad (2.73)$$

with critical masses given by

$$\begin{aligned} M^2_{s,k} &= -4\lambda^{(s)}_k - 4\lambda^{(s)}_{k+1} \Delta_{s+k,s} \frac{(k+s+2\epsilon_0)(k+s+\epsilon_0+\frac{3}{2})(k+2s+2\epsilon_0+1)}{(k+s+1)(k+2s+2\epsilon_0)(k+s+\epsilon_0+\frac{1}{2})} \\ &= -\sigma\left(4\epsilon_0 + 2s + (k+2s+2\epsilon_0+1)k\right) . \end{aligned} \quad (2.74)$$

These masses can be computed without the explicit usage of the expression for $\lambda^{(s)}_k$, by first rewriting (2.69) as $\nabla_a \Phi_{(s)} = i\mathrm{ac}_{P_a} \Phi_{(s)}$ and then using the Casimir relation $C_2[\mathfrak{g}] = C_2[\mathfrak{m}] - 2\sigma P^a \star P_a$, that is $-\mathrm{ac}_{P^a} \mathrm{ac}_{P_a} \Phi_{(s)} = \frac{\sigma}{2}(\widetilde{\mathrm{ad}}_{M_{AB}} \widetilde{\mathrm{ad}}_{M^{AB}} - \mathrm{ad}_{M_{ab}} \mathrm{ad}_{M^{ab}}) \Phi_{(s)}$. This yields

$$\nabla^2 \Phi_{a(s+k),b(s)} = \sigma\left(C_2[\mathfrak{g}|\ell] - C_2[\mathfrak{m}|(s+k,s)]\right) \Phi_{a(s+k),b(s)} , \quad (2.75)$$

with $C_2[\mathfrak{m}|(s+k,s)] = (s+k)(s+k+D-2) + s(s+D-4)$. Inserting the value of $C_2[\mathfrak{g}|\ell]$ given by (2.30) then gives (2.74).

[21] The consistency at higher orders in weak fields depends crucially on the contributions from $J_{(n)}$ with $n \geqslant 1$ and $P_{(n)}$ $n \geqslant 2$.

2.7 Harmonic expansion of the Weyl zero-form

The maximally symmetric coset geometries (2.5) can be embedded into Vasiliev's equations as exact solutions given by

$$\Phi = 0\,, \qquad A \;=\; A_{(2)} \;=\; \Omega \;\equiv\; L^{-1}\star dL\,, \tag{2.76}$$

where Ω is thus a flat $\mathfrak{g}$-connection parameterized by a coset element[22] $L = L(x)$. The leading order of the weak-field expansion around these solutions is a self-consistent set of linearized field equations that are invariant under abelian gauge transformations. The linearized zero-form constraint (2.69) is solved by

$$\Phi_{(s)} \;=\; L^{-1}\star S_{(s)}\star\pi(L)\,,\quad dS_{(s)} \;=\; 0\;. \tag{2.77}$$

Expanding $S_{(s)}$ in the twisted-adjoint compact-weight module $\mathcal{M}_{(s)}(\sigma,\sigma')$,

$$S_{(s)} \;=\; \sum_{\kappa} S^{(s)}_{\kappa} T^{(s)}_{\kappa}\;,\quad S^{(s)}_{\kappa}\in\mathbb{C}\,, \tag{2.78}$$

yields the *harmonic expansion* of the linearized Weyl tensors

$$\Phi_{(s)} \;=\; \sum_{\kappa} S^{(s)}_{\kappa} L^{-1}\star T^{(s)}_{\kappa}\star\pi(L) \;=\; \sum_{k=0}^{\infty} T_{a(s+k),b(s)}\sum_{\kappa} S^{(s)}_{\kappa} D^{(s);a(s+k),b(s)}_{\kappa}\,, \tag{2.79}$$

where the *generalized harmonic functions*

$$\begin{aligned} D^{(s);a(s+k),b(s)}_{\kappa} &= \mathcal{N}^{-1}_{s,k}\ \mathrm{Tr}\left[T^{a(s+k),b(s)}\star L^{-1}\star T^{(s)}_{\kappa}\star\pi(L)\right] \\ &= \mathcal{N}^{-1}_{s,k}\,{}_{12}\langle T_{a(s+k),b(s)}|L^{-1}|\kappa\rangle_{12}\;. \end{aligned} \tag{2.80}$$

The first equality follows from (B.18), while the second from (2.44), (2.48) and the overlap condition (2.46), which implies that $\pi(L)(1)|\mathbb{1}\rangle_{12} = L^{-1}(2)|\mathbb{1}\rangle_{12}$ so that L^{-1} acts in the diagonal representation where $M_{AB} = M_{AB}(1) + M_{AB}(2)$ and

$$|\kappa\rangle_{12} \;=\; T^{(s)}_{\kappa}(1)|\mathbb{1}\rangle_{12}\,,\qquad |T_{a(s+k),b(s)}\rangle \;=\; T_{a(s+k),b(s)}(1)|\mathbb{1}\rangle_{12}\;. \tag{2.81}$$

The harmonic functions obey the Bianchi identity (2.72) and the mass-shell condition (2.73).

In Euclidean signature, the bilinear forms $(\cdot,\cdot)_{\mathfrak{ho}}$ and $(\cdot,\cdot)_{\mathcal{T}}$ are *positive definite* for $\sigma = -1$ and $\sigma = +1$, respectively. Thus, the $\mathfrak{m}$-covariant expansion of Φ around H_D provides a unitarizable $\mathfrak{ho}$-module. In Lorentzian signature, the unitarizable $\mathfrak{ho}$-modules arise in Φ for both $\sigma = +1$ (AdS_D) and $\sigma = -1$ (dS_D) upon going to the corresponding compact bases of $\mathcal{T}$. The former case will be examined next and the latter case is examined in section 3.

[22] For example, in Lorentz-covariant stereographic coordinates one may take $e^a = -\frac{2\lambda dx^a}{h^2}$ and $\omega^{ab} = -\frac{4\sigma\lambda^2 x^{[a}dx^{b]}}{h^2}$ with $h = \sqrt{1-\sigma\lambda^2x^2}$, which arise from $L = \exp_\star(4i\xi\lambda x^a P_a) = f\exp(i\lambda g x^\mu \delta^a_\mu P_a)$ with $\xi = \frac{\mathrm{artanh}\sqrt{\frac{1-h}{1+h}}}{\sqrt{1-h^2}}$ and $f = f(h)$ and $g = g(h)$. In $D = 4,6$, where $\mathcal{T}_{(0)}$ is conformal, one has $f = \left[\frac{2h}{1+h}\right]^{\epsilon_0+\frac{1}{2}}$ and $g = \frac{4}{1+h}$ (see appendix F).

JHEP10(2008)022

2.8 Unitarity of the harmonic expansion on dS_D

The the real form $\mathfrak{so}(1,D)$ with Lorentz algebra $\mathfrak{so}(1,D-1)$ and transvections obeying $[P_a,P_b]_\star=-iM_{ab}$, splits into $\mathfrak{h}\Subset\mathfrak{l}$ where the maximal compact subalgebra $\mathfrak{h}=\mathfrak{so}(D)'$ is generated by $J_{mn}=(M_{rs},P_r)$ obeying $[J_{mn},J_{pq}]_\star=4i\delta_{[p|[n}J_{m]|q]}$, and $\mathfrak{l}=\mathrm{span}_{\mathbb{R}}K_m$ where $K_m=(M_{r0},P_0)$ obeys $[K_m,K_n]_\star=iJ_{mn}$. We can define the twisted-adjoint compact-weight module $\mathcal{M}(-,+)=\mathcal{M}^{(+)}(-,+)\oplus\mathcal{M}^{(-)}(-,+)$, where $\mathcal{M}^{(\pm)}(-,+)$ are $\mathfrak{ho}$-irreps that decompose under $\mathfrak{so}(D,1)$ into

$$\mathcal{M}^{(\pm)}(-,+)=\bigoplus_{s=0}^{\infty}\mathcal{M}^{(\pm)}_{(s)}(-,+)\,,\qquad \mathcal{M}^{(\pm)}_{(s)}(-,+)\;=\;\bigoplus_{k=0}^{\infty}\mathcal{M}^{(\pm)(s)}_{(s+k,s)}\,, \tag{2.82}$$

where $\mathcal{M}^{(\pm)}_{(s)}(-,+)$ consists of generalized elements in $\mathcal{T}_\ell$, $s=2\ell+2$, given by regular series expansions, and $\mathcal{M}^{(\pm)(s)}_{(s+k,s)}$ are traceless type-$(s+k,s)$ tensors of $\mathfrak{so}(D)'$. These decompose further under $\mathfrak{s}$ as $\mathcal{M}^{(\pm)(s)}_{(s+k,s)}=\bigoplus_{s+k\geqslant j_1\geqslant s\geqslant j_2\geqslant 0}\mathcal{M}^{(\pm)(s)}_{(s+k,s|j_1,j_2)}$ where the traceless type-(j_1,j_2) basis elements

$$T^{(\pm)(s)}_{(s+k,s|j_1,j_2)}=\sum_{n=0}^{\infty}f^{(\pm)(s)}_{(s+k,s|j_1,j_2);n}T^{(s)}_{(j_1,j_2);n}\,,\quad \left[T^{(s)}_{(j_1,j_2);n}\right]_{r(j_1),t(j_2)}=T_{0(n)\{r(j_1),t(j_2)\}0(s-j_2)}\,, \tag{2.83}$$

with $T_{a(m),b(n)}$ defined in (2.26) and $f^{(\pm)(s)}_{(s+k,s|j_1,j_2);n}\in\mathbb{R}$ determined by the above embedding conditions.[23] For example, using

$$\widetilde{P}_{\{r|}T_{0(n)|r(k)\}}=2T_{0(n)\{r(k+1)\}}+\frac{n(n-1)(n+k+1)(n+k+2\epsilon_0-1)}{8(n+k+\epsilon_0-\frac12)(n+k+\epsilon_0+\frac12)}T_{0(n-2)\{r(k+1)\}}\,, \tag{2.84}$$

for $s=0$ one finds that the generating functions $f^{(\pm)(0)}_{(k|k)}(z)=\sum_{n=0}^\infty f^{(\pm)(s)}_{(k|k);n}z^n$ of the "top" elements $T^{(\pm)(0)}_{(k|k)}$ are given up to overall constants by

$$f^{(+)(0)}_{(0|0)}(z)\;=\;{}_2F_3\left[\frac{k+\epsilon_0+\frac32}{2},\frac{k+\epsilon_0+\frac52}{2};\frac12,\frac{k+3}{2},\frac{k+2\epsilon_0+1}{2};-4z^2\right]\,, \tag{2.85}$$

$$f^{(-)(0)}_{(0|0)}(z)\;=\;z\,{}_2F_3\left[\frac{k+\epsilon_0+\frac32}{2},\frac{k+\epsilon_0+\frac52}{2};\frac32,\frac{k+4}{2},\frac{k+2\epsilon_0+2}{2};-4z^2\right]\,, \tag{2.86}$$

that simplify in $D=4$ to $f^{(+)(0)}_{(k|k)}(z)=\cos 4z$ and $f^{(-)(0)}_{(k|k)}(z)=\frac14\sin 4z$. The $\widetilde{\mathfrak{ho}}$ action on $T^{(\pm)(0)}_{(0|0)}$ fill out the spaces $\mathcal{M}^{(\pm)}(-,+)$. By a choice of normalization, the twisted-adjoint $\mathfrak{so}(D,1)$ representation matrix in $\mathcal{M}^{(\pm)}_{(s)}(-,+)$ takes the twisted-adjoint form $\widetilde{K}T^{(\pm)(s)}_{(s+k|s)}=2T^{(\pm)(s)}_{(s+k+1|s)}+2\lambda^{(s)}_k\mathbf{P}_{(s+k,s)}\left[\delta\otimes T^{(\pm)(s)}_{(s+k-1|s)}\right]$ with $\lambda^{(s)}_k$ given by (A.28). These

[23] An $\mathfrak{so}(D)'$-tensor of type-(s_1,s_2) decomposes under $\mathfrak{s}$ into type-(j_1,j_2) tensors with $s_1\geqslant j_1\geqslant s_2\geqslant j_2\geqslant 0$, and can thus be a (generalized) element of $\mathcal{T}_\ell$ for $s=2\ell+2$ with expansion of the form given in (2.26) only if $s_2=s$. For $D=4$ the type-(j_1,j_2) tensors with $j_2\geqslant 2$ are trivial, since an irreducible $\mathfrak{so}(N;\mathbb{C})$-tensor is trivial if the sum of the heights of the first two columns in its Young diagram exceeds N.

$\mathfrak{so}(D,1)$ modules are unitarizable in the bilinear inner product

$$(S,S')_{\mathcal{M}^{(\pm)}_{(s)}(-,+)} \equiv \frac{1}{\mathcal{N}^{(\pm)}_{(s)}}\mathrm{Tr}(\pi(S)\star S')\,, \tag{2.87}$$

defined by the analog of the prescription that we shall give in detail later on, below (3.84), with $\mathrm{Tr}(\pi(T^{(\pm)(s)}_{(s,s|s,s)})\star T^{(\pm)(s)}_{(s,s|s,s)}) = \mathcal{N}^{(\pm)}_{(s)}\mathbf{P}_{(s,s)}$. For example, for $s=0$ it follows from $\pi((T^{(\epsilon)(0)}_{(0|0)})^\dagger) = \pi(T^{(\epsilon)(0)}_{(0|0)}) = \epsilon T^{(\epsilon)(0)}_{(0|0)}$ and $\pi(\widetilde{\mathrm{ad}}_Q S)^\dagger) = -\widetilde{\mathrm{ad}}_{Q^\dagger}\pi(S^\dagger)$ that the real forms are given by $S^{(\epsilon)} = \sqrt{\epsilon}\sum_k i^k S^{(\epsilon)(0)}_{(k)} T^{(\epsilon)(0)}_{(k)}$ with $S^{(\epsilon)(0)}_{(k)}\in\mathbb{R}$, and hence[24]

$$\begin{aligned}(S,S')_{\mathcal{M}^{(\pm)}_{(0)}(-,+)} &= \pm\sum_{p,q}\left[S^{(\pm)(0)}_{(p)}\right]^{m(p)}\frac{2^{-p}}{\mathcal{N}^{(\pm)}_{(0)}}\mathrm{Tr}\left[S^{(\pm)(0)}_{(0)}\star(\widetilde{K})^p_{m(p)}(\widetilde{K})^q_{n(q)}S^{(\pm)(0)}_{(0)}\right]\left[S^{(\pm)(0)\prime}_{(q)}\right]^{n(q)}\\ &= \pm\sum_p N_p\, S^{(\pm)}_{(p)}\cdot S^{(\pm)(0)\prime}_{(p)}\,,\qquad N_p = \prod_{k=0}^{p}\lambda^{(0)}_k\,,\end{aligned} \tag{2.88}$$

with $(\widetilde{K})^p_{m(p)} = \widetilde{K}_{\{m_1}\cdots\widetilde{K}_{m_p\}}$, that are manifestly positive or negative definite.

3. On the harmonic expansion on AdS_D

We first define the twisted-adjoint compact-weight modules $\mathcal{M}(+,+|\mu)$ of $\mathfrak{so}(2,D-1)$ consisting of basis elements $T^{(s)}_{e;(j_1,j_2)}$ with energy $e\in\mathbb{Z}+\mu$ and spin (j_1,j_2), given by *regular* series expansions in the Lorentz covariant basis, and corresponding to harmonic functions that are *finite* in the interior of $CAdS_D$. In the case of $\mu=0$ we propose that the even and odd submodules $\mathcal{M}^{(\pm)}(+,+|0)$ are generated by $\widetilde{\mathfrak{ho}}$ from the *static ground states* $T^{(0)}_{0;(0)}$ and $T^{(0)}_{0;(1)}$. Then, in terms of *one-sided* $\mathcal{A}$-modules $\mathcal{S}^{(s)}_{e;(j_1,j_2)}$ and their duals, we identify the Flato-Fronsdal factorization of a subsector of $\mathcal{M}(+,+|0)$, and also the complete factorization of $\mathcal{M}^{(+)}(+,+|0)$ in terms of $\mathcal{S}^{(0)}_{0;(0)}$ and its dual. These fill wedges in compact-weight space and we shall refer to them as *angletons*, as opposed to the singletons that fill single lines. The angletons contain (one-sided) $\mathfrak{so}(D-1)$-submodules with *negative spin*, and the factorization is given by a direct product modulo an equivalence relation (see (3.32)). We then proceed to identifying the standard composite-massless spaces $\mathfrak{D}^\pm(\pm(s+2\epsilon_0);(s))$ as invariant subspaces of $\mathcal{M}_{(s)}(+,+|0)$. Their complements $\mathcal{W}_{(s)}$ are (composite-massless) *lowest-spin modules* in the sense that they are unbounded in ordinary weight space, i.e. the energy and spin eigenvalues are not bounded from above nor below, while the spins (j_1,j_2) obey $j_1\geqslant j_2\geqslant 0$. We propose that the Tr on $\mathcal{A}$ induces norms (i.e. bilinear forms with definite signatures) on $\mathcal{W}_{(s)}$, as we shall verify explicitly in the case of $\mathcal{W}^{(+)}_{(0)}$. Finally, we describe the map from $\mathcal{M}^{(s)}(+,+)$ to $\mathcal{T}_\ell$ as a decomposition of the reflector $|\mathbb{1}\rangle_{12}$. As by-product, we find a natural generalization of the Flato-Fronsdal

[24] The harmonic expansion of $(\nabla^2-M^2)\phi=0$ in dS_D yields a compact-weight module $\mathcal{M}_{(0)}(M^2)$ with $C_2[\mathfrak{so}(D,1)|M^2]=-M^2$ and representation matrix where $\lambda^{(0)}_k(M^2)=\frac{k}{8(k+\epsilon_0+\frac{1}{2})}(k^2+2\epsilon_0 k-1-2\epsilon_0+M^2)$. For $M^2>0$ this module is irreducible and unitarizable. If $M^2=M^2_p\equiv-(p+2\epsilon_0+1)(p-1)$ then it follows from $\lambda^{(0)}_k(M^2_p)>0$ for $k>p$, and $\lambda^{(0)}_p(M^2_p)=0$, that $\mathcal{M}_{(0)}(M^2_p)$ contains the unitarizable invariant subspace $\mathcal{M}_{(0)}(M^2_p)'=\bigoplus_{k\geqslant p}\mathcal{M}^{(0)}_{(k)}(M^2_p)$.

formula whereby the adjoint representation $\mathfrak{ho}$ is identified as the direct product between singletons and anti-singletons (see (3.174)).

3.1 Twisted-adjoint compact-weight module

The twisted-adjoint $\mathfrak{so}(2)\oplus\mathfrak{so}(D-1)$-covariant modules $\mathcal{M}(+,+|\mu)$, $\mu\in[0,1]$, are defined by

$$\mathcal{M}(+,+|\mu) = \bigoplus_{s=0}^{\infty}\mathcal{M}_{(s)}(+,+|\mu)\,,\qquad \mathcal{M}_{(s)}(+,+|\mu) \;=\; \bigoplus_{\substack{e-\mu\in\mathbb{Z}\\ j_1\geqslant s\geqslant j_2\geqslant 0}} \mathbb{C}\otimes T^{(s)}_{e;(j_1,j_2)}\;, \tag{3.1}$$

where the $\mathfrak{g}$ submodules $\mathcal{M}_{(s)}(+,+|\mu)$ consist of *regular* elements $T^{(s)}_{e;(j_1,j_2)}$ with spin (j_1,j_2) and energy $e\in\mathbb{R}$, that is

$$\left[T^{(s)}_{e;(j_1,j_2)}\right]_{r(j_1),t(j_2)} = \sum_{n=0}^{\infty} f^{(s)}_{e;(j_1,j_2);n}\left[T^{(s)}_{(j_1,j_2);n}\right]_{r(j_1),t(j_2)}\;, \tag{3.2}$$

where $T^{(s)}_{(j_1,j_2);n}\in\mathcal{A}$ are the same as in (2.83), and the generating functions $f^{(s)}_{e;(j_1,j_2)}(z)=\sum_{n=0}^{\infty} f^{(s)}_{e;(j_1,j_2);n}z^n$ are determined *uniquely* from

$$\mathrm{ac}_E T^{(s)}_{e;(j_1,j_2)} = e\,T^{(s)}_{e;(j_1,j_2)}\;,\qquad f^{(s)}_{e;(j_1,j_2);0} \;=\; 1\,, \tag{3.3}$$

as can be seen from (A.26) which implies $\mathrm{ac}_E(T^{(s)}_{(j_1,j_2);n}) = \lambda^{(s)}_{(j_1,j_2);n}T^{(s)}_{(j_1,j_2);n+1} + \lambda'^{(s)}_{(j_1,j_2):n}T^{(s)}_{(j_1,j_2);n-1}$ where the coefficients are non-vanishing except $\lambda'^{(s)}_{0;(j_1,j_2)} = 0$. It also follows that $f^{(s)}_{e;(j_1,j_2);n}\in\mathbb{R}$ which together with $(T^{(s)}_{(j_1,j_2);n})^\dagger = T^{(s)}_{(j_1,j_2);n}$ implies that $(T^{(s)}_{e;(j_1,j_2)})^\dagger = T^{(s)}_{e;(j_1,j_2)}$. Moreover, $\pi(T^{(s)}_{e;(j_1,j_2)}) = (-1)^{j_1-s}T^{(s)}_{-e;(j_1,j_2)}$, that is $\pi:\ \mathcal{M}_{(s)}(+,+|\mu)\longrightarrow\mathcal{M}_{(s)}(+,+|1-\mu)$, and $f^{(s)}_{-e;(j_1,j_2)}(z) = f^{(s)}_{e;(j_1,j_2)}(-z)$. The ideal relations imply that

$$\widetilde{L}^{\pm}_{r}\left[T^{(s)}_{e;(s,j_2)}\right]_{rt(s-1),u(j_2)} = 0 \qquad \text{for } j_1=s\geqslant 1 \text{ and } j_2<s\,, \tag{3.4}$$

$$\mathbf{P}_{\{j_1,j_2,1\}}\left[\widetilde{L}^{\pm}_{u}\left[T^{(s)}_{e;(j_1,j_2)}\right]_{r(j_1),t(j_2)}\right] = 0 \qquad \text{for } j_2\geqslant 1\,, \tag{3.5}$$

and from $\widetilde{C}_{2n}[\mathfrak{g}]\left(T^{(s)}_{e;(j_1,j_2)}\right) = \sum_{p=0}^{\infty} f^{(s)}_{e;(j_1,j_2)}\frac{1}{2}\widetilde{M}_{A_1}{}^{A_2}\cdots\widetilde{M}_{A_{2n}}{}^{A_1}\left(T^{(s)}_{p;(j_1,j_2)}\right)$ it follows that

$$\widetilde{C}_{2n}[\mathfrak{g}|\mathcal{M}_{(s)}] = C_{2n}[\ell]\;, \tag{3.6}$$

where $s=2\ell+2$ and $C_2[\ell]$ and $C_4[\ell]$ are given in (2.30). The space $\mathcal{M}(+,+|\mu)$ and its subspaces $\mathcal{M}_{(s)}(+,+|\mu)$ decompose under the $\widetilde{\mathfrak{ho}}$ and $\widetilde{\mathfrak{g}}$ actions into *even and odd submodules* $\mathcal{M}(+,+|\mu)=\mathcal{M}^{(+)}(+,+|\mu)\oplus\mathcal{M}^{(-)}(+,+|\mu)$ where

$$\mathcal{M}^{(\pm)}(+,+|\mu) = \bigoplus_{\substack{e;\,(j_1,j_2)\\ e-\mu+j_1+j_2\\ =\frac{1}{2}(1\mp 1)\ \mathrm{mod}\ 2}} \mathbb{C}\otimes T^{(s)}_{e;(j_1,j_2)}\;. \tag{3.7}$$

The Weyl zero-form Φ (obeying $\Phi^\dagger = \pi(\Phi)$) can be expanded as

$$\Phi = \int d\mu\ \Phi(\mu)\,, \qquad \Phi(\mu) \ = \ \sum_s \Phi_{(s)}(\mu)\,, \tag{3.8}$$

$$\Phi_{(s)}(\mu) = \sum_{\substack{e-\mu\in\mathbb{Z}\\ j_1\geqslant s\geqslant j_2\geqslant 0}} \ \sum_{k=j_1-s}^{\infty} T_{a(s+k),b(s)} S^{(s)}_{e;(j_1,j_2)}(\mu) D^{(s);a(s+k),b(s)}_{e;(j_1,j_2)}\ , \tag{3.9}$$

where $S^{(s)}_{e;(j_1,j_2)}(\mu)\in\mathbb{C}$ obey the reality condition

$$\left(S^{(s)}_{e;(j_1,j_2)}(\mu)\right)^* = (-1)^{j_1-s} S^{(s)}_{-e;(j_1,j_2)}(1-\mu)\ , \tag{3.10}$$

and the generalized harmonic functions ($k\geqslant j_1-s$)

$$\begin{aligned}\left[D^{(s);a(s+k),b(s)}_{e;(j_1,j_2)}\right]_{r(j_1),t(j_2)} &= \mathcal{N}^{-1}_{s,k}\ \mathrm{Tr}\left[T^{a(s+k),b(s)}\star L^{-1}\star\left[T^{(s)}_{e;(j_1,j_2)}\right]_{r(j_1),t(j_2)}\star\pi(L)\right]\\ &= \mathcal{N}^{-1}_{s,k}\,{}_{12}\langle T_{a(s+k),b(s)}|L^{-1}|(s);e;(j_1,j_2)\rangle_{12;r(j_1),t(j_2)}\,,\end{aligned} \tag{3.11}$$

where $|(s);e;(j_1,j_2)\rangle_{12} = T^{(s)}_{e;(j_1,j_2)}(1)\star|\mathbb{1}\rangle_{12}$. At $L=1$ the overlaps are finite and given by

$$\begin{aligned}\left[D^{(s);a(s+k),b(s)}_{e;(j_1,j_2)}\right]_{r(j_1),t(j_2)}|_{L=1} &= \delta^{\{a(s+k),b(s)\}_D}_{\{0(n)\{r(j_1),t(j_2)\}_{D-1}0(s-j_2)\}_D} f^{(s)}_{e;(j_1,j_2);n}\,,\\ n &= s+k-j_1\,,\end{aligned} \tag{3.12}$$

where $\{\cdots\}_D$ and $\{\cdots\}_{D-1}$, respectively, denote $\mathfrak{so}(D)$ and $\mathfrak{so}(D-1)$ traceless Young projections.

In what follows we shall focus on the case $\mu=0$, and we shall therefore write $\mathcal{M}\equiv\mathcal{M}(+,+|0)$, $\mathcal{M}^{(\pm)}\equiv\mathcal{M}^{(\pm)}(0)$ and $\mathcal{M}^{(\pm)}_{(s)}\equiv\mathcal{M}^{(\pm)}_{(s)}(0)$.

3.2 Static ground states

We propose that $\mathcal{M}^{(\pm)}_{(s)}$ are generated by $\widetilde{\mathcal{U}}[\mathfrak{g}]$ from the elements with $e=0$ and minimal j_1+j_2, namely the *static ground states*

$$s=0:\quad T^{(0)}_{(\pm)} \ = \ T^{(0)}_{0;(\sigma_\pm)}\ ; \qquad\qquad s>0\quad T^{(s)}_{(\pm)} \ = \ T^{(s)}_{0;(s,\sigma_\pm)}\ , \tag{3.13}$$

where $\sigma_\pm = (1\mp 1)/2$. Furthermore, we propose that $T^{(s)}_{(\pm)}$ with $s>0$ are generated by $\widetilde{\mathcal{U}}[\mathfrak{ho}]$ action from $T^{(0)}_{(\pm)}$, so that

$$\mathcal{M}^{(\pm)}_{(s)} = \mathcal{U}[\widetilde{\mathfrak{g}}]T^{(s)}_{(\pm)}\,, \qquad \mathcal{M}^{(\pm)} \ = \ \mathcal{U}[\widetilde{\mathfrak{ho}}]T^{(0)}_{(\pm)}\ . \tag{3.14}$$

As shown in appendix F, the generating functions of $T^{(0)}_{(\pm)}$ are given by

$$f^{(0)}_{0;(0)}(z) = \sum_{p=0}^{\infty}\frac{(4z)^{2p}(\epsilon_0+\frac{3}{2})_{2p}}{(2)_{2p}(2\epsilon_0+1)_{2p}} = {}_2F_3\left(\frac{2\epsilon_0+3}{4},\frac{2\epsilon_0+5}{4};\frac{3}{2},\epsilon_0+\frac{1}{2},\epsilon_0+1;4z^2\right)\ , \tag{3.15}$$

$$f^{(0)}_{0;(1)}(z) = \sum_{p=0}^{\infty}\frac{(\epsilon_0+\frac{5}{2})_{2p}\,z^{2p}}{p!(2)_p(\epsilon_0+1)_p(\epsilon_0+2)_p} = {}_2F_3\left(\frac{2\epsilon_0+5}{4},\frac{2\epsilon_0+7}{4};2,\epsilon_0+1,\epsilon_0+2;4z^2\right)\ . \tag{3.16}$$

In $D=4$ these functions are[25]

$$f^{(0)}_{0;(0)}(z) = \frac{\sinh 4z}{4z}\,, \qquad f^{(0)}_{0;(1)}(z) \;=\; \frac{3}{16z^2}\left(\cosh 4z - \frac{\sinh 4z}{4z}\right)\,. \tag{3.17}$$

The $\widetilde{\mathfrak{g}}$-action in $\mathcal{M}$ obeys relations of the form

$$\left(\widetilde L^\pm_t \widetilde L^\mp_t - \mu^{(s)}_{\pm e;(j_1,j_2)}\right) T^{(s)}_{e;(j_1,j_2)} \;=\; \left(\widetilde x^\pm \widetilde x^\mp - \mu'^{(s)}_{\pm e;(j_1,j_2)}\right) T^{(s)}_{e;(j_1,j_2)} \;=\; 0\,, \tag{3.18}$$

$$\left(\widetilde x^\pm \widetilde L^\mp_{\{r_1}\widetilde L^\mp_{r_2\}} - \mu''^{(s)}_{\pm e;(j_1,j_2)} \widetilde L^\pm_{\{r_1}\widetilde L^\mp_{r_2\}}\right) T^{(s)}_{e;(j_1,j_2)} = 0\,, \qquad \widetilde x^\pm \;=\; \widetilde L^\pm_r \widetilde L^\pm_r\,, \tag{3.19}$$

where $\mu^{(s)}_{e;(j_1,j_2)}$, $\mu'^{(s)}_{e;(j_1,j_2)}$,$\mu''^{(s)}_{e;(j_1,j_2)} \in \mathbb{R}$. To factor out these relations we first write $\mathcal{M}^{(+)}_{(s)} = \mathcal{M}^{(+)>}_{(s)} \cup \mathcal{M}^{(+)0}_{(s)} \cup \mathcal{M}^{(+)<}_{(s)}$ with

$$\mathcal{M}^{(+)\gtrless}_{(s)} = \{T^{(s)}_{e;(j_1,j_2)}\;:\; \pm e > j_1+j_2-s\}\,, \quad \mathcal{M}^{(+)0}_{(s)} = \{T^{(s)}_{e;(j_1,j_2)}\;:\; |e| \leqslant j_1+j_2-s\}\,. \tag{3.20}$$

Thus, according to our proposal, there exist *non-vanishing* coefficients $\mathcal{C}^{(s)}_{e;(j_1,j_2)}$ such that

$$\mathcal{M}^{(+)\gtrless}_{(s)}\;:\; \left[T^{(s)}_{e;(j_1,j_2)}\right]_{r(j_1),t(j_2)} \;=\; \mathcal{C}^{(s)}_{e;(j_1,j_2)} (\widetilde x^\pm)^p \widetilde L^{\pm(j_1-s)}_{\{r(j_1-s)} \widetilde L^{\pm(j_2)}_{t(j_2)} \left[T^{(s)}_{0;(s,0)}\right]_{r(s)\}}\,, \tag{3.21}$$

for $p_\pm = \frac{1}{2}(\pm e + s - j_1 - j_2)$ and where $\widetilde L^{\pm(n)}_{\{r(n)} = \widetilde L^+_{\{r_1} \cdots \widetilde L^+_{r_n\}}$, and such that

$$\mathcal{M}^{(+)0}_{(s)}\;:\; \left[T^{(s)}_{e;(j_1,j_2)}\right]_{r(j_1),t(j_2)} = \mathcal{C}^{(s)}_{e;(j_1,j_2)} (\widetilde L^{+(q_+)} \widetilde L^{-(q_-)})_{\{r_1\cdots r_{j_1-s} t_1 \cdots t_{j_2}} \left[T^{(s)}_{0;(s,0)}\right]_{r(s)\}}\,, \tag{3.22}$$

for $q_\pm = \frac{1}{2}(j_1+j_2-s\pm e)$ and where $(\widetilde L^{+(m)}\widetilde L^{-(n)})_{r_1\cdots r_{m+n}} = \widetilde L^+_{\{r_1}\cdots \widetilde L^+_{r_m} \widetilde L^-_{r_{m+1}} \cdots \widetilde L^-_{r_{m+n}\}}$. In particular, for $s=0$ one has

$$T^{(0)}_{e+2;(0)} = \mathcal{C}_e \widetilde x^+\, T^{(0)}_{e;(0)}\,, \qquad \mathcal{C}_e \;=\; -\frac{1}{(e+2\epsilon_0)(e+2)}\,, \tag{3.23}$$

$$\widetilde L^-_r T^{(0)}_{e;(0)} = \mathcal{C}'_e \widetilde L^+_r T^{(0)}_{e-2;(0)}\,, \qquad \mathcal{C}'_e \;=\; \frac{1}{\mathcal{C}'_{2-e}} \;=\; -\frac{(e-2\epsilon_0)(e-2)}{e(e+2\epsilon_0-2)}\,, \tag{3.24}$$

which implies $\mu^{(0)}_{e;(0)} = (e-2\epsilon_0)(e-2)$, $\mu'^{(0)}_{e;(0)} = (e-2\epsilon_0)(e-2)e(e+2\epsilon_0-2)$, and

$$(\widetilde L^+_r + \widetilde L^-_r) f(\widetilde x^+) T^{(0)}_{e;(0)} \;=\; (D_e f)(\widetilde x^+) T^{(0)}_{e;(0)}\,, \tag{3.25}$$

for differentiable functions $f(\widetilde x^+)$ and with

$$D_e \;=\; 4\widetilde x^+ \frac{d^2}{d(\widetilde x^+)^2} + 4(e-\epsilon_0)\frac{d}{d\widetilde x^+} + 1 + \frac{(e-2\epsilon_0)(e-2)}{\widetilde x^+}\,. \tag{3.26}$$

For even $s = 2p \geqslant 2$ the static ground states are of the form

$$\left[T^{(2p)}_{0;(2p)}\right]_{r(2p)} \;=\; \sum_{n=0}^{p-1} \xi_{2p;n} \widetilde L^+_{\{r_1} \widetilde L^-_{r_2} \cdots \widetilde L^+_{r_{2n-1}} \widetilde L^-_{r_{2n}} \widetilde Q_{r(2p-2n)\}} T^{(0)}_{0;(0)}\,, \tag{3.27}$$

[25]The function $f^{(0)}_{0;(0)}(z)$ corresponds to the twisted-adjoint element representing the static and rotationally invariant scalar-field profile discussed in [24].

JHEP10(2008)022

$$\left[T^{(2p)}_{0;(2p,1)}\right]_{r(2p),s} = \sum_{n=0}^{p-1} \xi'_{2p;n} \widetilde{L}^+_{\{r_1} \widetilde{L}^-_{r_2} \cdots \widetilde{L}^+_{r_{2n-1}} \widetilde{L}^-_{r_{2n}} \widetilde{Q}_{r(2p-2n)} \left[T^{(0)}_{0;(1)}\right]_{s\}} , \tag{3.28}$$

for some $\xi_{2p;n}, \xi'_{2p;n} \in \mathbb{R}$ and $Q_{r(2n)} = L^+_{\{r_1} \star L^-_{r_2} \star \cdots \star L^+_{r_{2n-1}} \star L^-_{r_{2n}\}} \in \mathfrak{ho}$. Similarly, for odd spin $s = 2p+1 \geqslant 3$ one has

$$\left[T^{(2p+1)}_{0;(2p+1)}\right]_{r(2p+1)} = \sum_{n=0}^{p-1} \xi_{2p+1;n} \widetilde{L}^+_{\{r_1} \widetilde{L}^-_{r_2} \cdots \widetilde{L}^+_{r_{2n-1}} \widetilde{L}^-_{r_{2n}} \widetilde{Q}_{r(2p-2n)} \left[T^{(1)}_{0;(1)}\right]_{r_{2p+1}\}} , \tag{3.29}$$

$$\left[T^{(2p+1)}_{0;(2p+1,1)}\right]_{r(2p+1),s} = \sum_{n=0}^{p-1} \xi'_{2p+1;n} \widetilde{L}^+_{\{r_1} \widetilde{L}^-_{r_2} \cdots \widetilde{L}^+_{r_{2n-1}} \widetilde{L}^-_{r_{2n}} \widetilde{Q}_{r(2p-2n)} \left[T^{(1)}_{0;(1,1)}\right]_{r_{2p+1},s\}} , \tag{3.30}$$

for some $\xi_{2p+1;n}, \xi'_{2p+1;n} \in \mathbb{R}$, and where $T^{(1)}_{(\pm)}$ in their turn can be generated from the scalar static ground states. For example, to generate $T^{(1)}_{0;(1)}$ from $T^{(0)}_{0;(1)}$ one may use $\widetilde{\mathrm{ad}}_{EM_{rs}} \left[T^{(0)}_{0;(1)}\right]_t = \left\{EM_{rs}, \left[T^{(0)}_{0;(1)}\right]_t\right\} = \delta_{t[s} \left[T^{(1)}_{0;(1)}\right]_{r]}$, as follows from $EM_{rs} = E \star M_{rs} = M_{rs} \star E$ and $E \star \left[T^{(0)}_{0;(1)}\right]_r = -\left[T^{(0)}_{0;(1)}\right]_r \star E = \frac{1}{2}\mathrm{ad}_E \left[T^{(0)}_{0;(1)}\right]_r = -\frac{i}{2}\left[T^{(1)}_{0;(1)}\right]_r$. The generation of $T^{(1)}_{0;(1,1)}$ from $T^{(0)}_{0;(0)}$ is more involved since $E \star T^{(0)}_{0;(0)} = \frac{1}{2}\mathrm{ad}_E T^{(0)}_{0;(0)} = 0$. For example, two $\widetilde{\mathfrak{g}}$ transformations send $T^{(0)}_{0;(0)}$ into $T^{(0)}_{0;(2)}$, which $\widetilde{\mathrm{ad}}_{EM_{rs}}$ maps to $T^{(1)}_{0;(2)}$, from which two $\widetilde{\mathfrak{g}}$ transformations lead down to $T^{(1)}_{0;(1,1)}$.

3.3 Factorization in terms of singletons and angletons

The element $T^{(s)}_{e;(j_1,j_2)}$ gives rise to separate $\mathcal{A}$ left and right modules

$$\mathcal{S}^{(s)}_{e;(j_1,j_2)} = \mathcal{A} \star T^{(s)}_{e;(j_1,j_2)} , \qquad \mathcal{S}^{(s)*}_{e;(j_1,j_2)} = T^{(s)}_{e;(j_1,j_2)} \star \mathcal{A} , \tag{3.31}$$

which are subspaces of $\mathcal{M}^{(\pm)}(+,+|\mu)$ for $e - \mu + j_1 + j_2 = \frac{1}{2}(1 \mp 1)$ mod 2. According to (3.14) an element $S \in \mathcal{M}^{(\pm)}$ can be written as $S = \sum_{X,X' \in \mathcal{A}} X \star T^{(0)}_{(\pm)} \star X'$. Thus, the twisted-adjoint compact-weight modules can be factorized as follows:

$$\mathcal{M}^{(\pm)} = (\mathcal{S}^{(\pm)} \otimes \mathcal{S}^{(\pm)*})/\sim , \qquad \mathcal{S}^{(\pm)} = \mathcal{S}^{(0)}_{0;(\sigma_\pm)} , \quad \mathcal{S}^{(\pm)*} = \mathcal{S}^{(0)*}_{0;(\sigma_\pm)} , \tag{3.32}$$

where we shall refer to the factors as *angletons*, and the equivalence relation reads

$$(X \star T^{(0)}_{(\pm)}) \otimes (T^{(0)}_{(\pm)} \star Y) \sim (X' \star T^{(0)}_{(\pm)}) \otimes (T^{(0)}_{(\pm)} \star Y') \;\Leftrightarrow\; X \star T^{(0)}_{(\pm)} \star Y = X' \star T^{(0)}_{(\pm)} \star Y' . \tag{3.33}$$

The ideal relations $V_{AB} \approx 0$ and $C_2[\mathfrak{g}|\mathcal{M}^{(\pm)}] \approx -\epsilon_0(\epsilon_0 + 2)$ imply that

$$L^+_r \star L^+_r \approx L^-_r \star L^-_r \approx 0 , \qquad M_{rs} \star L^\pm_s \approx i(\epsilon_0 \pm E) \star L^\pm_r , \tag{3.34}$$

$$M_{\{r_1}{}^t \star M_{r_2\}t} \approx \frac{1}{2}\{L^+_{\{r_1}, L^-_{r_2\}}\}_\star , \qquad L^+_{[r} \star L^-_{s]} \approx i(1-E) \star M_{rs} , \tag{3.35}$$

$$\frac{1}{2} M^{rs} \star M_{rs} \approx E \star E - \epsilon_0^2 , \qquad \{L^+_r, L^-_r\}_\star \approx 4(E \star E + \epsilon_0) , \tag{3.36}$$

which together with $V_{ABCD} \approx 0$ show that $\mathcal{A}$ has the compact basis

$$\mathcal{A} = \bigoplus_{\substack{e \in \mathbb{Z},\ n \geqslant 0 \\ j_1 \geqslant j_2 \geqslant 0 \\ |e| \leqslant j_1 - j_2}} \mathbb{C} \otimes M_{e;(j_1,j_2);n} \,, \tag{3.37}$$

where $\left[M_{e;(j_1,j_2);n}\right]_{r(j_1),s(j_2)} = L^{+(p_+)}_{\{r(p_+)} L^{-(p_-)}_{r(p_-)} M^{(j_2)}_{r(j_2)s(j_2)\}} E^n$, $p_\pm = \frac{1}{2}(j_1 \pm e - j_2)$, has adjoint energy e and spin (j_1, j_2), and $L^{\pm(p)}_{r(p)} \equiv L^\pm_{r_1} \cdots L^\pm_{r_p}$, $M^{(p)}_{r(p)s(p)} \equiv M_{r_1 s_1} \cdots M_{r_p s_p}$. Since $T^{(0)}_{e;(0)}$ are series expansions in $T_{0(n)}$, which are $\star$-polynomials in E of order n (as follows from (F.4)), it follows that $[M_{0;(j_1,j_2);p}, T^{(0)}_{e;(0)}]_\star = 0$. In particular, $\mathrm{ad}_E T^{(0)}_{e;(0)} = 0$, which together with $\mathrm{ac}_E T^{(0)}_{e;(0)} = e T^{(0)}_{e;(0)}$ and (3.36) yields

$$E \star T^{(0)}_{e;(0)} = T^{(0)}_{e;(0)} \star E \;=\; \frac{e}{2} T^{(0)}_{e;(0)} \,, \tag{3.38}$$

$$C_2[\mathfrak{s}] \star T^{(0)}_{e;(0)} = \nu_e(\nu_e + 2\epsilon_0) T^{(0)}_{e;(0)} \,, \qquad \nu_e \;=\; \frac{|e|}{2} - \epsilon_0 \,, \tag{3.39}$$

$$\{L^+_r, L^-_r\}_\star \star T^{(0)}_{e;(0)} = \mu_e T^{(0)}_{e;(0)} \,, \qquad \mu_e \;=\; e^2 + 4\epsilon_0 \,. \tag{3.40}$$

Thus $\mathcal{S}^{(0)}_{e;(0)}$ is spanned by the elements $M_{e';(j_1',j_2');0} \star T^{(0)}_{e;(0)}$ as can be seen by re-ordering $M_{e';(j_1',j_2');p} = \sum_{n=0}^p \mathcal{C}^{p,n}_{e';(j_1',j_2')} M_{e';(j_1,j_2);0} \star E^n$, where $\mathcal{C}^{p,n}_{e';(j_1',j_2')}$ are finite coefficients, which implies that $M_{e';(j_1',j_2');p} \star T^{(0)}_{e;(0)} = \sum_{n=0}^p \mathcal{C}^{p,n}_{e';(j_1',j_2')} (e')^n M_{e';(j_1',j_2');0} \star T^{(0)}_{e;(0)}$. In particular, the even angleton

$$\mathcal{S}^{(+)} = \bigoplus_{\substack{j_1 \geqslant j_2 \geqslant 0 \\ |e| < j_1 - j_2}} \mathbb{C} \otimes T_{e;(j_1,j_2)} \,, \qquad T_{e;(j_1,j_2)} \;\equiv\; M_{e;(j_1,j_2);0} \star T^{(0)}_{0;(0)} \,, \tag{3.41}$$

where the basis elements $T_{e;(j_1,j_2)}$ carry a representation of the left action, viz. $M_{e;(j_1,j_2);n} \star T_{e';(j_1',j_2')} = \sum_{e'';(j_1'',j_2'')} \mathcal{C}^{e'';(j_1'',j_2'')}_{e;(j_1,j_2);n|e';(j_1',j_2')} T_{e'';(j_1'',j_2'')}$, and also twisted-adjoint energy and spin $(e;(j_1,j_2))$, that is $T_{e;(j_1,j_2)} = \sum_{s=j_2}^{j_1} \mathcal{C}^{(s)}_{e;(j_1,j_2)} T^{(s)}_{e;(j_1,j_2)}$. The one-dimensionality of the compact weights $(0;(0))$, $(0;(1,1))$ and $(2;(1,1))$, implies that there exist finite coefficients α, α' and α'' such that

$$L^\pm_r \star T^{(0)}_{(+)} \star L^\pm_r \;=\; \alpha T^{(0)}_{(+)} \,, \quad L^\pm_{[r} \star T^{(0)}_{(+)} \star L^\pm_{s]} \;=\; \alpha' M_{rs} \star T^{(0)}_{(+)} \,, \tag{3.42}$$

$$M_{rs} \star L^\pm_t \star T^{(0)}_{(+)} \star L^\pm_t \;=\; \alpha'' L^\pm_{[r} \star T^{(0)}_{(+)} \star L^\mp_{s]} \,, \tag{3.43}$$

where on the right-hand sides $M_{rs} \star T^{(0)}_{0;(0)} = \frac{4\epsilon_0}{(2\epsilon_0+1)(2\epsilon_0+2)} \left[T^{(1)}_{0;(1,1)}\right]_{r,s}$ and $L^\pm_t \star T^{(0)}_{(+)} \star L^\pm_t = 2\epsilon_0 T^{(0)}_{2;(0)}$, as can be seen using (F.23) and (3.23), respectively. The remaining combinations involving one left and one right $\star$-multiplication of ladder operators are non-degenerate, and we conclude that the equivalence relation (3.33) is generated by

$$(M_{0;(j_1,j_2);0} \star T^{(0)}_{(+)}) \otimes T^{(0)}_{(+)} \sim T^{(0)}_{(+)} \otimes (T^{(0)}_{(+)} \star M_{0;(j_1,j_2);0}) \,, \tag{3.44}$$

$$(L^\pm_r \star T^{(0)}_{(+)}) \otimes (T^{(0)}_{(+)} \star L^\pm_r) \sim \alpha T^{(0)}_{(+)} \otimes T^{(0)}_{(+)} \,, \tag{3.45}$$

JHEP10(2008)022

$$(L^{\pm}_{[r}\star T^{(0)}_{(+)})\otimes(T^{(0)}_{(+)}\star L^{\pm}_{s]})\sim\alpha'(M_{rs}\star T^{(0)}_{(+)})\otimes T^{(0)}_{(+)}\,, \tag{3.46}$$

$$(M_{rs}\star L^{\pm}_{t}\star T^{(0)}_{(+)})\otimes(T^{(0)}_{(+)}\star L^{\pm}_{t})\sim\alpha''(L^{\pm}_{[r}\star T^{(0)}_{(+)})\otimes(T^{(0)}_{(+)}\star L^{\mp}_{s]})\,. \tag{3.47}$$

The left $\mathfrak{g}$-module $\mathcal{S}^{(0)}_{e;(0)}$ contains the left $\mathfrak{s}$-submodule

$$\mathcal{S}(\nu_e)\equiv\bigoplus_{n=0}^{\infty}\bigoplus_{p=0}^{\frac{n}{2}}\mathbb{C}\otimes\mathbf{P}_{\{n,n-2p\}}M^{(n)}_{r(n)s(n)}\star T^{(0)}_{e;(0)}\,,\qquad \nu_e\;=\;\frac{|e|}{2}-\epsilon_0\,, \tag{3.48}$$

where the traceless type-$(n,n-2p)$ projections $\mathbf{P}_{\{n,n-2p\}}$ incorporate the ideal relations $C_2[\mathfrak{s}]\approx C_2[\mathfrak{s}|\nu_e]=\nu_e(\nu_e+2\epsilon_0)$ given in (3.39) and $\mathbf{P}_{\{2,1,1\}}M_{[r_1s}M_{t]r_2}\approx 0$. It follows that $\nu_e\in\mathbb{Z}+[\frac{1}{2}(e+D)]$, and we say that $\mathcal{S}(\nu_e)$ carries half-integer or integer one-sided spins, respectively. Moreover, if $|e|<2\epsilon_0$ then $C_2[\mathfrak{s}|\nu_e]<0$ and say that $\mathcal{S}(\nu_e)$ carries negative one-sided spins.

From (F.24) it follows that

$$\text{all } D:\qquad \mathfrak{D}^{\pm}\;=\;\mathfrak{D}^{+}\oplus\mathfrak{D}^{-}\;\equiv\;\mathcal{A}\star T^{(0)}_{\pm 2\epsilon_0;(0,0)}\star\mathcal{A}\,, \tag{3.49}$$

$$D\neq 5:\qquad \mathfrak{D}'^{\pm}\;=\;\mathfrak{D}'^{+}\oplus\mathfrak{D}'^{-}\;\equiv\;\frac{\mathcal{A}\star T^{(0)}_{\pm 2;(0,0)}\star\mathcal{A}}{(\mathcal{A}\star T^{(0)}_{\pm 2;(0,0)}\star\mathcal{A})\cap\mathfrak{D}^{\pm}}\,, \tag{3.50}$$

are two-sided $\mathcal{A}$ submodules in $\mathcal{M}(+,+|0)$, and we note that $(\mathcal{A}\star T^{(0)}_{\pm 2;(0,0)}\star\mathcal{A})\cap\mathfrak{D}^{\pm}$ is non-vanishing iff D is odd. There is an analog of $\mathfrak{D}'$ for $D=5$ to be defined below. In $\mathfrak{D}^{+}$ one has that if $e=2\epsilon_0+2n$ with $n=0,1,2,\dots$ then the one-sided spins are positive integers. From (F.24) and the fact that $\mathcal{C}_{2\epsilon_0+2n}$ in (3.23) are non-vanishing, it follows that $L^{-}_{r_1}\star\cdots\star L^{-}_{r_{2n+1}}\star T^{(0)}_{2\epsilon_0+2n;(0)}=0$. Thus, viewed as a one-sided $\mathcal{A}$ module

$$\mathcal{S}^{(0)}_{2\epsilon_0+2n;(0)}\simeq\mathfrak{D}(\epsilon_0;(0))\quad\text{for } n=0,1,2,\dots\;. \tag{3.51}$$

Moreover, viewed two-sidedly, the element $T^{(0)}_{2\epsilon_0+2n;(0)}\simeq|\epsilon_0+n;(n)\rangle_{r(n)}{}^{r(n)}\langle\epsilon_0+n;(n)|$, which yields the isomorphism

$$\mathfrak{D}^{+}\simeq\mathfrak{D}_0\otimes\mathfrak{D}_0^{*}\,, \tag{3.52}$$

underlying the enveloping-algebra analog of the Flato-Fronsdal formula (3.64), that we shall discuss in the next Subsection. Likewise, in $D=4$ we identify the spinor singleton

$$D=4:\mathcal{S}^{(0)}_{2+2n;(0)}\;\simeq\;\mathfrak{D}\left(1;\left(\frac{1}{2}\right)\right)\quad\text{for } n=0,1,2,\dots\,, \tag{3.53}$$

leading to the isomorphism $T^{(0)}_{2+2n;(0)}\simeq|1+n;(n+\frac{1}{2})\rangle^{i,r(n)}{}_{i,r(n)}\langle 1+n;(n+\frac{1}{2})|$ underlying the enveloping-algebra version of the Flato-Fronsdal formula for the 4D spinor singleton:

$$D=4:\mathfrak{D}'^{+}\;\simeq\;\mathfrak{D}_{\frac{1}{2}}\otimes\mathfrak{D}^{*}_{\frac{1}{2}}\,, \tag{3.54}$$

where King's rule rules out the elements with twisted-adjoint energies between 2 and the composite values. The scalar singletons and the spinor singletons in $D=4$ have natural extensions by negative spins described in appendix E although their role in the factorization of $\mathcal{M}^{(\pm)}$ and its submodules is unclear at the moment.

3.4 Lowest-weight and lowest-spin submodules

3.4.1 Admissibility analysis

Lowest-weight and highest-weight $\widetilde{\mathfrak{g}}$ submodules in $\mathcal{M}_{(s)}$ correspond to the solutions of

$$\widetilde{L}_r^- T^{(s)}_{e;(j_1,j_2)} = L_r^- \star T^{(s)}_{e;(j_1,j_2)} - T^{(s)}_{e;(j_1,j_2)} \star L_r^+ = 0 \ . \tag{3.55}$$

If this holds, then $C_2[\mathfrak{g}]$ and $C_4[\mathfrak{g}]$ are on the one hand given by (C.1) and (C.2), and on the other hand by (3.6). This leads to the necessary conditions

$$\begin{aligned} x+y+z &= x_0+y_0\,, \\ x(x+\Delta)+y(y+\Delta')+z(z+\Delta'') &= x_0(x_0+\Delta)+y_0(y_0+\Delta')\,, \end{aligned} \tag{3.56}$$

where $x = e(e-D+1)$, $y = j_1(j_1+D-3)$, $z = j_2(j_2+D-5)$ and $\Delta = \frac{1}{2}(D-1)(D-2)$, $\Delta' = \frac{1}{2}(D-3)(D-4)-1$ and $\Delta'' = \frac{1}{2}(D-5)(D-6)-2$, and finally $x_0 = e_0(e_0-D+1)$, $y_0 = s(s+D-3)$ and $e_0 = s+D-3$. Moreover, combining (3.55) with (3.4) yields

$$e = s+D-3-\frac{j_2}{s} \qquad \text{for } j_1 = s \geqslant 1 \text{ and } j_2 < s\ . \tag{3.57}$$

The combination of (3.55) and (3.5) yields yet another necessary condition, valid for $j_2 \geqslant 1$, which we shall not need here.

If $j_2 = 0$ then $z = 0$, and (3.56) has two roots: $(x,y) = (x_0,y_0)$ and (y_0+2-D, x_0+D-2). The first root corresponds to $j_1 = s$ and $e = s+D-3$ or $e = 2-s$. The latter energy level is ruled out for $s \geqslant 1$ due to the condition (3.57). The second root corresponds to $j_1 = s-1$, which is ruled out for all s, and $j_1 = 4-D-s$, which is ruled for all s except $s = 0$ in $D = 4$ where it coincides with the first root (which is thus a double root in $D = 4$). Thus, the admissible lowest-weight states with $j_2 = 0$ are

$$j_1 = s\,, \quad e = s+2\epsilon_0 \qquad \text{and} \quad j_1 = s = 0\,, \quad e = 2\,, \tag{3.58}$$

where we note the two-fold degeneracy of the root $j_1 = s = 0$, $e = 2$ for $D = 5$. If $j_2 = s \geqslant 1$, which requires $D \geqslant 5$, then the only admissible state is

$$j_1 = j_2 = s = 1\,, \qquad e = 2\ . \tag{3.59}$$

In the remaining case of $D \geqslant 5$ and $s > j_2 \geqslant 1$, we have no conclusive statements to make, though we have not found any admissible root here.

3.4.2 Case of $s = 0$

The two admissible roots for $s = 0$ are realized by the lowest-weight states with generating functions

$$f^{(0)}_{2\epsilon_0;(0)}(z) = {}_1F_1\left(\epsilon_0+\frac{3}{2};2;-4z\right), \qquad f^{(0)}_{2;(0)}(z) = {}_1F_1\left(\epsilon_0+\frac{3}{2};2\epsilon_0;-4z\right), \tag{3.60}$$

taking the following particularly simple form in $D = 4$:

$$f^{(0)}_{1;(0)}(z) = e^{-4z}\,, \qquad f^{(0)}_{2;(0)}(z) = (1-4z)e^{-4z}\ . \tag{3.61}$$

For $D = 2p+5$, $p = 1,2,3,\ldots$, the Harish-Chandra module $\mathfrak{C}(2;(0))$ contains the singular vector $|2\epsilon_0;(0)\rangle = (x^+)^p|2;(0)\rangle$ and $\mathfrak{C}(2;(0)) = \mathfrak{D}(2;(0)) \supsetplus \mathfrak{D}(2\epsilon_0;(0))$ where $\mathfrak{D}(2;(0)) = \mathfrak{C}(2;(0))/\mathfrak{N}(2;(0))$ is the scalar p-lineton[26] and $\mathfrak{D}(2\epsilon_0;(0)) = \mathfrak{N}(2;(0)) = \mathfrak{C}(2\epsilon_0;(0))$ is the composite-massless lowest-weight space. In the enveloping-algebra realization, the module $\mathfrak{C}(2;(0)) \simeq \widetilde{\mathcal{U}}[\mathfrak{g}]T^{(0)}_{2;(0)}$ and

$$T^{(0)}_{2\epsilon_0;(0)} = \mathcal{C}_{2\epsilon_0-2}\cdots\mathcal{C}_2(\widetilde{x}^+)^p\, T^{(0)}_{2;(0)} \quad \text{for } D = 2p+5,\ p = 1,2,\ldots\ . \tag{3.62}$$

Thus, the scalar modules $\mathcal{M}^{(\pm)}_{(0)}$ contain the ideals:

$$\begin{array}{lcl} D = 4,6,\ldots\colon \mathcal{I}^{(+)}_{(0)} &=& (1+\pi)\mathfrak{D}^+(2;(0))\,, \quad \mathcal{I}^{(-)}_{(0)} \;=\; (1+\pi)\mathfrak{D}^+(2\epsilon_0;(0))\,, \\ D = 5 \qquad\ \ \colon \mathcal{I}^{(+)}_{(0)} &=& (1+\pi)\mathfrak{D}^+(2;(0))\,, \\ D = 7,9,\ldots\colon \mathcal{I}^{(+)}_{(0)} &=& (1+\pi)\left[\mathfrak{D}^+(2;(0)) \supsetplus \mathfrak{D}^+(2\epsilon_0;(0))\right]\ . \end{array} \tag{3.63}$$

3.4.3 Case of $s \geqslant 1$ and the Flato-Fronsdal formula

From (F.24) it follows that $L^-_r(\xi)|2\epsilon_0;(0)\rangle_{12} = M_{rs}(\xi)|2\epsilon_0;(0)\rangle_{12} = 0$ for $\xi = 1,2$ which yields an enveloping-algebra analog of the Flato-Fronsdal formula:

$$|s+2\epsilon_0;(s)\rangle_{12;r(s)} = f_{(s)}f_{r(s)}(1,2)|2\epsilon_0;(0)\rangle_{12}\,, \tag{3.64}$$

where $f_{(s)}$ is a normalization fixed by (3.3) and $f_{r(s)}$ the composite operator[27]

$$f_{r(s)}(1,2) = (-1)^s f_{r(s)}(2,1) = \sum_{k=0}^{s} f_{s;k}(L^+_{\{r_1}\cdots L^+_{r_k})(1)(L^+_{r_{k+1}}\cdots L^+_{r_s\}})(2)\,, \tag{3.65}$$

$$f_{s;k} = (-1)^s f_{s;s-k} = \binom{s}{k}\frac{(1-s-\epsilon_0)_k}{(\epsilon_0)_k}\ . \tag{3.66}$$

Indeed, applying $\langle \mathbb{1}^*|_{23}$ to (3.64) yields the lowest-weight elements:

$$\left[T^{(s)}_{s+2\epsilon_0;(s)}\right]_{r(s)} = f_{(s)}\sum_{k=0}^{s}(-1)^{s-k}f_{s;k}L^+_{\{r_1}\star\cdots\star L^+_{r_k}\star T^{(0)}_{2\epsilon_0;(0)}\star L^-_{r_{k+1}}\star\cdots\star L^-_{r_s\}}\,. \tag{3.67}$$

Likewise, in $D = 4$ the lowest-weight spaces $\mathfrak{D}(2;(0))$ and $\mathfrak{D}(2;(1,1)) \simeq \mathfrak{D}(2;(1))$ are the first two levels of the tower of composite massless lowest-weight spaces $\mathfrak{D}(s+1;(s,1)) \simeq$

[26]For $e_0 = \epsilon_0 - (p-1)$, $p = 1,2,\ldots$, the Harish-Chandra module $\mathfrak{C}(e_0;(0)) \supset \mathfrak{N}(e_0+2p;(0))$ with singular vector given by $(x^+)^p|e_0;(0)\rangle$. Thus $\mathfrak{D}(e_0;(0))$, that we shall refer to as the scalar p-lineton, consists of p lines in weight space, viz.

$$\mathfrak{D}(e_0;(0)) = \bigoplus_{k=0}^{p-1}\bigoplus_{n=0}^{\infty}|e_0+2k+n;(n)\rangle\,, \quad |e_0+2k+n;(n)\rangle_{r(n)} = L^+_{r_1}\cdots L^+_{r_n}(x^+)^k\mid e_0;(0)\rangle\ .$$

In particular, the 1-lineton coincides with the ordinary singleton.

[27]The condition $(L^-_r(1)+L^-_r(2))|s+2\epsilon_0;(s)\rangle_{12;r(s)} = 0$ is equivalent to $a_k f_{s;k} + a_{s-k+1}f_{s;k-1} = 0$ with $a_k = 2k(k+\epsilon_0-1)$ implying $f_{s;k} = (-1)^s f_{s;s-k} = (-1)^k\frac{a_{s-k+1}\cdots a_s}{a_k\cdots a_1}f_{s;0}$, that becomes (3.66) for $f_{s;0} = 1$.

$\mathfrak{D}(s+1;(s))$ contained in the tensor product $\mathfrak{D}_{\frac{1}{2}} \otimes \mathfrak{D}_{\frac{1}{2}}$ of two spinor singletons. We expect that all these lowest-weight spaces are realized in $\mathcal{M}^{(+)}$:

$$D\!=\!4: \left[T^{(s)}_{s+1;(s)}\right]_{r(s)} \!=\! f'_{(s)}\epsilon_{tu\{r_1}\sum_{k=1}^{s} f'_{s;k} L^+_{r_2} \star \cdots \star L^+_{r_k} \star M_{tu} \star T^{(0)}_{2;(0)} \star L^-_{r_{k+1}} \star \cdots \star L^-_{r_s\}}\ , \tag{3.68}$$

where $f'_{(s)}$ and $f'_{s;k}$ are fixed by (3.3) and (3.55), respectively.

For $D \geqslant 6$ it follows from $C_2[\mathfrak{s}|(1-\epsilon_0)] < 0$ that the $\mathfrak{s}$ submodule $\mathcal{S}(1-\epsilon_0)$ defined in (3.48) is infinite-dimensional, thus consisting of the elements $M_{\{r_1 t_1} \star \cdots M_{r_j t_j\}} \star T^{(0)}_{2;(0)} = \mathcal{C}^{(j)}_{2;(j,j)} \left[T^{(j)}_{2;(j,j)}\right]_{r(j),s(j)}$, $j = 0,1,2,\ldots$, for non-vanishing $\mathcal{C}^{(j)}_{2;(j,j)}$, since $M_{\{r_1}{}^{t} \star M_{r_2\}t} \star T^{(0)}_{2;(0)} \approx \frac{1}{2}\{L^+_{\{r_1}, L^-_{r_2\}}\}_\star \star T^{(0)}_{2;(0)} = 0$. Thus $\widetilde{L}^-_u T^{(j)}_{2;(j,j)} = 0$ for $j \geqslant 0$ and $D \geqslant 6$, and by analytical continuation also for $D = 5$, that is

$$\widetilde{L}^-_u T^{(s)}_{2;(s,s)} = 0 \quad \text{for } s = 1,2,\ldots \text{ and } D \geqslant 5\,, \tag{3.69}$$

while $D = 4$ falls under King's rule. For example, $\mathcal{M}^+_{(1)}$ contains the generalized Verma module $\mathfrak{C}'(2;(1,1)) = \frac{\mathfrak{C}(2;(1,1))}{\mathcal{I}[V]}$, which is isomorphic to $\widetilde{\mathcal{U}}[\mathfrak{g}]T^{(1)}_{2;(1,1)}$ modulo (3.4) and (3.5). For $D = 3+2p$, $p = 1,2,\ldots$, the lowest-weight state of $\mathfrak{D}(1+2\epsilon_0;(1)) = \mathfrak{D}(1+2p;(1))$ is a singular vector in $\mathfrak{C}'(2;(1,1))$, viz.[28] $\widetilde{L}^-_t(\widetilde{x}^+)^{p-1}\widetilde{L}^+_s T^{(1)}_{2;(1,1);u,s} \approx 0$. Factoring out the ideal $\mathfrak{N}'(2;(1,1)) \simeq \mathfrak{D}(1+2\epsilon_0;(1))$ from $\mathfrak{C}'(2;(1,1))$ yields the spin-1 p-lineton

$$\mathfrak{D}'(2;(1,1)) = \frac{\mathfrak{C}'(2;(1,1))}{\mathfrak{N}'(2;(1,1))} \quad \text{for } D = 3+2p,\ p = 1,2,\ldots\,, \tag{3.70}$$

occupying p lines in compact weight space. In $D = 5$ it is the ordinary spin-1 singleton, that is $\mathfrak{D}'(2;(1,1)) = \mathfrak{D}(2;(1,1))$.

Thus, in summary, the sectors $\bigoplus_{s=1}^{\infty} \mathcal{M}^{(\pm)}_{(s)}$ contain the ideals:

$$D = 4: \mathcal{I}^{(+)} = (1+\pi)\bigoplus_{s=1}^{\infty}\mathfrak{D}^+(1+s;(s,1))\,, \tag{3.71}$$

$$\mathcal{I}^{(-)} = (1+\pi)\bigoplus_{s=1}^{\infty}\mathfrak{D}^+(1+s;(s))\,, \tag{3.72}$$

$$D = 5,7,\ldots: \mathcal{I}^{(+)} = (1+\pi)\bigoplus_{s=1}^{\infty}[\mathfrak{D}^+(2\epsilon_0+s;(s)) \oplus \mathfrak{D}^+(2;(s,s))]\,, \tag{3.73}$$

$$D = 6,8,\ldots: \mathcal{I}^{(+)} = (1+\pi)\bigoplus_{s=1}^{\infty}\mathfrak{D}^+(2;(s,s))\,, \tag{3.74}$$

$$\mathcal{I}^{(-)} = (1+\pi)\bigoplus_{s=1}^{\infty}\mathfrak{D}^+(2\epsilon_0+s;(s))\ . \tag{3.75}$$

[28] At the level of the Harish-Chandra module $\mathfrak{C}(2;(1,1))$,

$$L^-_t x^{p-1} L^+_s|2\rangle_{s,u} = -4(p-1)x^{p-2}L^+_u L^+_s|2\rangle_{s,t} + 2(p-1)(2p+5-D)x^{p-2}L^+_t L^+_s|2\rangle_{s,u} + 2(5-D)x^{p-1}|2\rangle_{t,u}\,,$$

where $|2\rangle_{s,u} \equiv |2;(1,1)\rangle_{s,u}$. For $D = 2p+3$ the (tu)-projection vanishes and the $[tu]$-projection equals $-6(D-5)L^+_s L^+_{[s}|2\rangle_{t,u]}$ that vanishes in $\mathfrak{C}(2;(1,1))$ only if $D = 5$, while it vanishes weakly in $\mathfrak{C}'(2;(1,1))$ for all D.

3.4.4 Summary of the indecomposable structure of twisted-adjoint module

In summary, the compact twisted-adjoint module has the indecomposable structure

$$\mathcal{M} = \mathcal{W} \supsetplus \mathfrak{D}' \supsetplus \mathfrak{D} \ , \tag{3.76}$$

where $\mathfrak{D} = \mathfrak{D}^+ \oplus \mathfrak{D}^-$ and $\mathfrak{D}' = \mathfrak{D}'^+ \oplus \mathfrak{D}'^-$ have the substructures

$$\text{all } D : \mathfrak{D} \ \simeq \ (1+\pi)(\mathfrak{D}_0^+ \otimes \mathfrak{D}_0^+) \ \simeq \ (1+\pi) \bigoplus_{s=0}^{\infty} \mathfrak{D}^+(s+2\epsilon_0;(s)) \,, \tag{3.77}$$

$$D=4 : \mathfrak{D}' \ \simeq \ (1+\pi)(\mathfrak{D}_{\frac{1}{2}}^+ \otimes \mathfrak{D}_{\frac{1}{2}}^+) \ \simeq \ (1+\pi) \bigoplus_{s=0}^{\infty} \mathfrak{D}^+(s+2\epsilon_0+\delta_{s,0};(s,1)) \,, \tag{3.78}$$

$$D=5 : \mathfrak{D}' \ \simeq \ (1+\pi) \bigoplus_{s=1}^{\infty} \mathfrak{D}^+(2;(s,s)) \,, \tag{3.79}$$

$$D \geqslant 6 : \mathfrak{D}' \ \simeq \ (1+\pi) \left(\mathfrak{D}^+(2;(0)) \oplus \bigoplus_{s=0}^{\infty} \mathfrak{D}'^+(2;(s,s)) \right) \ . \tag{3.80}$$

Moreover, splitting into even and odd parts we have

$$D=4,6,\cdots : \qquad \mathcal{M}^{(+)} = \mathcal{W}^{(+)} \supsetplus \mathfrak{D}' \,, \qquad \mathcal{M}^{(-)} = \mathcal{W}^{(-)} \supsetplus \mathfrak{D} \,, \tag{3.81}$$

$$D=5,7,\cdots : \qquad \mathcal{M}^{(-)} = \mathcal{W}^{(-)} \,, \qquad \mathcal{M}^{(+)} = \mathcal{W}^{(+)} \supsetplus \mathfrak{D}' \supsetplus \mathfrak{D} \ . \tag{3.82}$$

The lowest-spin spaces $\mathcal{W}^{(\pm)}$ do not contain any lowest-weight nor highest-weight states. If $\mathcal{I}^{(\pm)}_{(0)}$ is non-trivial then $\mathcal{W}^{(\pm)}$ consists of the "wedge" $\mathcal{M}^{(\pm)0}$ defined in (3.22) plus a finite number of energy levels in $\mathcal{M}^{(\pm)\gtrless}$ for each j_1. In particular, one has

$$\mathcal{W}^{(+)}_{(0)} = \bigoplus_{|e| \leqslant j} \mathbb{C} \otimes T^{(0)}_{e;(j)} \ . \tag{3.83}$$

3.5 Inner products, real forms and unitarity

The non-polynomial nature of the compact basis elements $T^{(s)}_{e;(j_1,j_2)}$, viewed as generalized elements of $\mathcal{T}$, together with the indecomposable structure of $\mathcal{M}$ imply that the bilinear inner product $(S,S')_{\mathcal{T}} = Tr[\pi(S) \star S']$ induces inequivalent inner products in the different sectors $\mathfrak{V} = \mathfrak{D}, \mathfrak{D}', \mathcal{W}^{(\pm)}$ of $\mathcal{M}$:

$$(S,S')_{\mathfrak{V}} = \frac{1}{\mathcal{N}_{\mathfrak{V}}}(S,S')_{\mathcal{T}} \ , \qquad \mathcal{N}_{\mathfrak{V}} \ = \ (T_{\mathfrak{V}}, T_{\mathfrak{V}})_{\mathcal{T}} \ , \tag{3.84}$$

where $T_{\mathfrak{V}}$ denotes $T^{(0)}_{\pm 2\epsilon_0;(0)}$, $T^{(0)}_{\pm 2;(0)}$ and $T^{(0)}_{0;(\sigma_\pm)}$, respectively, and $\mathcal{N}_{\mathfrak{V}}$ is factored out according to the following *prescription*:

(i) expand S and S' in the bases generated by the $\widetilde{\mathfrak{ho}}$ action on $T_{\mathfrak{V}}$;

(ii) use (2.60), i.e. $\mathrm{Tr}[\pi(\widetilde{\mathrm{ad}}_Q S) \star S'] = -\mathrm{Tr}[\pi(S) \star \widetilde{\mathrm{ad}}_Q S']$ for $Q \in \mathfrak{ho}$ and $S, S' \in \mathcal{M}$ (which amounts to assuming $\mathrm{Tr}[X \star S] = \mathrm{Tr}[S \star X]$ for $X \in \mathcal{A}$ and general $S \in \mathcal{M}$) to write $(S,S')_{\mathcal{T}} = (T_{\mathfrak{V}}, \widetilde{Q}(S,S') T_{\mathfrak{V}})_{\mathcal{T}}$ for some $Q(S,S') \in \mathfrak{ho}$;

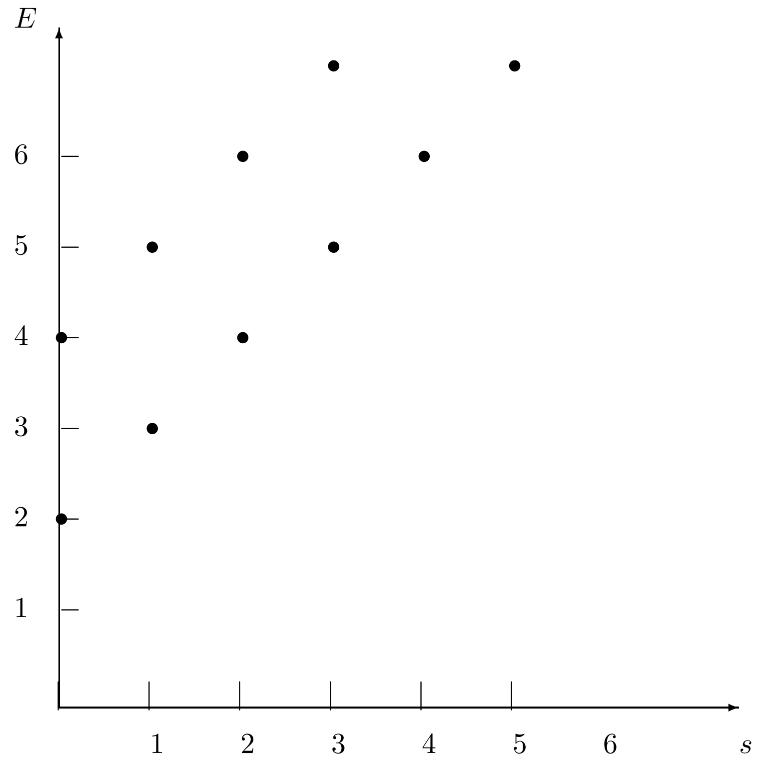

Figure 2: The $(\mathfrak{so}(2) \oplus \mathfrak{so}(D-1))$-types arising in the scalar 2-lineton in $D = 9$.

(iii) expand $\widetilde{Q}(S, S')T_{\mathfrak{Y}} = \sum_{s;e;(j_1,j_2)} \mathcal{C}^{(s)}_{e;(j_1,j_2)}(S, S')T^{(s)}_{e;(j_1,j_2)} = \mathcal{C}_{\mathfrak{Y}}T_{\mathfrak{Y}} + \cdots$, where $\mathcal{C}^{(s)}_{e;(j_1,j_2)}(S, S') \in \mathbb{C}$ are finite by construction, and declare $(S, S')_{\mathfrak{Y}}$ to be the coefficient of $T_{\mathfrak{Y}}$, i.e.

$$(S, S')_{\mathfrak{Y}} = \mathcal{C}_{\mathfrak{Y}}(S, S') \ . \tag{3.85}$$

By construction $(\cdot, \cdot)_{\mathfrak{Y}}$ is symmetric and $\mathfrak{ho}$-invariant; in particular, if $\widetilde{E}S_e = \{E, S_e\}_\star = eS_e$ idem $S'_{e'}$ then

$$(S_e, S'_{e'})_{\mathfrak{Y}} = \delta_{e+e',0}(S_e, S'_{e'})_{\mathfrak{Y}} \ . \tag{3.86}$$

Thus, on $\mathcal{M}^{(\pm)}$ we have the bilinear forms $(S, S')_{\mathcal{M}^{(\pm)}} = \mathcal{C}_{(\pm)}(S, S')$ where $\mathcal{C}_{(\pm)}(S, S')$ are the coefficients of $T_{(\pm)}$ in $-\widetilde{Q}(S)\widetilde{Q}(S')T_{(\pm)}$ with $\widetilde{Q}(S)$ and $\widetilde{Q}(S') \in \mathfrak{ho}$ defined by $S = \widetilde{Q}(S)T_{(\pm)}$ and $S' = \widetilde{Q}(S')T_{(\pm)}$. These bilinear forms split under(3.81) and (3.82) into a non-degenerate inner product on $\mathcal{W}^{(\pm)}$ and trivial bilinear forms on $\mathfrak{D}'$ and $\mathfrak{D}$, where the non-degenerate inner products are instead defined by $(S, S')_{\mathfrak{D}'} = \frac{1}{\mathcal{N}_2}Tr[\pi(S) \star S']$ and $(S, S')_{\mathfrak{D}} = \frac{1}{\mathcal{N}_{2\epsilon_0}}Tr[\pi(S) \star S']$. If S and S' obey the twisted-adjoint reality condition, i.e. $S^\dagger = \pi(S)$ idem S', then their inner product is a real number, i.e.

$$(S, S')_{\mathcal{M}^{(\pm)}} = \frac{1}{\mathcal{N}_{(\pm)}}\mathrm{Tr}[S^\dagger \star S'] = \frac{1}{\mathcal{N}_{(\pm)}}\mathrm{Tr}[(S'^\dagger \star S] = \left((S, S')_{\mathcal{M}^{(\pm)}}\right)^* , \tag{3.87}$$

$$(S,S')_{\mathfrak{D}} = \frac{1}{\mathcal{N}_{2\epsilon_0}}\mathrm{Tr}[S^\dagger \star S'] = \frac{1}{\mathcal{N}_{2\epsilon_0}}\mathrm{Tr}[(S')^\dagger \star S] = \big((S,S')_{\mathfrak{D}}\big)^* . \tag{3.88}$$

To examine their signatures we use the following bases:

$$S_{\mathcal{W}^{(\pm)}} = \sum_{s=0}^{\infty}\sum_{m,n=0}^{\infty} S^{(\pm)(s)}_{m,n}(\widetilde{L}^+)^m(\widetilde{L}^-)^n\widetilde{Q}_s T^{(0)}_{(\pm)}\,, \tag{3.89}$$

$$S_{\mathfrak{D}} = \sum_{s=0}^{\infty}\sum_{m}^{\infty}\left[S^{(s)}_m(\widetilde{L}^+)^m\widetilde{R}_s T^{(0)}_{2\epsilon_0;(0)} + \overline{S}^{(s)}_m(\widetilde{L}^-)^m\pi(\widetilde{R}_s T^{(0)}_{2\epsilon_0;(0)})\right] , \tag{3.90}$$

where $S^{(s)}_{m,n}, S^{(s)}_m, \overline{S}^{(s)}_m \in \mathbb{C}$ and $Q_s, R_s \in \mathfrak{ho}$ such that $\widetilde{Q}_s T^{(0)}_{(\pm)} = T^{(s)}_{(\pm)}$ and $\widetilde{R}_s T^{(0)}_{2\epsilon_0;(0)} = T^{(s)}_{s+2\epsilon_0;(s)}$. From $(L^\pm_r)^\dagger = L^\mp_r$ and $\pi(\widetilde{\mathrm{ad}}_Q S)^\dagger = -\widetilde{\mathrm{ad}}_{Q^\dagger}\pi(S^\dagger)$ it follows that

$$\mathcal{W}^{(\pm)}, \qquad s=0: \qquad (S^{(\pm)(0)}_{m,n})^* = (-1)^{\frac{1\mp 1}{2}}(-1)^{m+n}S^{(\pm)(0)}_{n,m}\,, \tag{3.91}$$

$$\mathcal{W}^{(\pm)}, \qquad s>0: \qquad (S^{(\pm)(s)}_{m,n})^* = (-1)^{m+n}S^{(\pm)(s)}_{n,m}\,, \tag{3.92}$$

$$\mathfrak{D}, \qquad s\geqslant 0: \qquad (S^{(s)}_m)^* = (-1)^m\overline{S}^{(s)}_m\,, \tag{3.93}$$

where the additional phase factor for $W^{(-)}_{(0))}$ arises from $\pi((T^{(0)}_{(\pm)})^\dagger) = (-1)^{\frac{1\mp1}{2}}T^{(0)}_{(\pm)}$. Thus

$$(S_{\mathcal{W}^{(\pm)}}, S'_{\mathcal{W}^{(\pm)}})_{\mathcal{M}} = \sum_{s=0}^{\infty}\sum_{m,n;m',n'=0}^{\infty}(S^{(\pm)(s)}_{m,n})^* N^{(\pm)(s)}_{m,n;m',n'}S^{(\pm)(s)\prime}_{m',n'}\,, \tag{3.94}$$

$$(S_{\mathfrak{D}}, S'_{\mathfrak{D}})_{\mathfrak{D}} = \sum_{s=0}^{\infty}\sum_{m,m'=0}^{\infty}\left((S^{(s)}_m)^* M^{(s)}_{m,m'}S^{(s)\prime}_{m'} + (S^{(s)\prime}_{m'})^* M^{(s)}_{m',m}S^{(s)}_m\right) , \tag{3.95}$$

$$\text{where} \qquad N^{(\pm)(s)}_{m,n;m',n'} = \left(T^{(s)}_\pm, (\widetilde{L}^-)^m(\widetilde{L}^+)^n(\widetilde{L}^+)^{m'}(\widetilde{L}^-)^{n'}T^{(s)}_\pm\right)_{\mathcal{W}^{(\pm)}}\,, \tag{3.96}$$

$$M^{(s)}_{m,m'} = \left(T^{(s)}_{-(s+2\epsilon_0);(s)}, (\widetilde{L}^-)^m(\widetilde{L}^+)^{m'}T^{(s)}_{s+2\epsilon_0;(s)}\right)_{\mathfrak{D}} . \tag{3.97}$$

More explicitly, using (3.21) we have $M^{(s)}_{m,n}((j_1,j_2)|(j_1',j_2')) = \delta_{m,n}\delta_{j_1,j_1'}\delta_{j_2,j_2'}M^{(s)}((j_1,j_2);p))$ where $p = \frac{1}{2}(m+s-j_1-j_2) \geqslant 0$ and $\left[M^{(s)}((j_1,j_2);p)\right]^{r'(j_1),t'(j_2)}_{r(j_1),t(j_2)}$ is given by

$$\left(T^{(s)}_{-(s+2\epsilon_0);(s);\{r(s)}, (\widetilde{x}^-)^p\widetilde{L}^{-(j_1-s)}_{r(j_1-s)}\widetilde{L}^{-(j_2)}_{t(j_2)\}}(\widetilde{x}^+)^p\widetilde{L}^{+(j_1-s)}_{\{r'(j_1-s)}\widetilde{L}^{+(j_2)}_{t'(j_2)}T^{(s)}_{s+2\epsilon_0;(s);r'(s)\}}\right)_{\mathfrak{D}} . \tag{3.98}$$

This matrix and hence the inner product $(\cdot,\cdot)_{\mathfrak{D}}$ is positive definite.[29]

The matrix $N^{(+)(s)}$ can be expanded using (3.21) and (3.22) as follows

$$N^{(+)(s)}_{m,n;m',n'}((j_1,j_2)|(j_1',j_2')) = \delta_{m,m'}\delta_{n,n'}\delta_{m+n,j_1+j_2-s}\delta_{j_1,j_1'}\delta_{j_2,j_2'}N^{(+)(s)}((j_1,j_2);p,q)\,, \tag{3.99}$$

[29]The complex space $\mathfrak{D}^+(s+2\epsilon_0;(s))\oplus\mathfrak{D}^-(s+2\epsilon_0;(s))$ has positive definite antilinear-linear inner product $\widehat{M}(e;(j_1,j_2)|e';(s_1',s_2')) = (|e;(j_1,j_2)\rangle)^\dagger\,|e';(s_1',s_2')\rangle = \begin{bmatrix} M & 0 \\ 0 & M\end{bmatrix}$ with M given by (3.97).

where $p = \frac{1}{2}(m - n + s - j_1 - j_2)$ with $n = 0$ for $p > 0$, and $q = \frac{1}{2}(n - m + s - j_1 - j_2)$ with $m = 0$ for $q > 0$, and $\left[N^{(+)(s)}((j_1, j_2); p, q)\right]^{r'(j_1),t'(j_2)}_{r(j_1),t(j_2)}$ is given by

$$\begin{cases} \left(T^{(s)}_{\pm;\{r(s)}, \left[(\widetilde{L}^-)^m(\widetilde{L}^+)^n\right]_{r(j_1-s),t(j_2)\}} \left[(\widetilde{L}^+)^m(\widetilde{L}^-)^n\right]_{\{r'(j_1-s),t'(j_2)} T^{(s)}_{\pm;r'(s)\}}\right)_{\mathcal{M}} & \text{for } p, q \leqslant 0 \\ \left(T^{(s)}_{\pm;\{r(s)}, (\widetilde{x}^-)^p(\widetilde{L}^-)^m_{r(j_1-s),t(j_2)\}} (\widetilde{x}^+)^p(\widetilde{L}^+)^m_{\{r'(j_1-s),t'(j_2)} T^{(s)}_{\pm;r'(s)\}}\right)_{\mathcal{M}} & \text{for } p > 0 \\ \left(T^{(s)}_{\pm;\{r(s)}, (\widetilde{x}^+)^q(\widetilde{L}^+)^n_{r(j_1-s),t(j_2)\}} (\widetilde{x}^-)^q(\widetilde{L}^-)^n_{\{r'(j_1-s),t'(j_2)} T^{(s)}_{\pm;r'(s)\}}\right)_{\mathcal{M}} & \text{for } q > 0 \end{cases} .$$

For $s = 0$ it follows from (3.83) that $N^{(+)(0)}_{m,n;m',n'}((j_1)|(j'_1)) = \delta_{m,m'}\delta_{n,n'}\delta_{,j_1}\delta_{j_1,s'_1}N^{(+)(0)}((j); p, q)$ where $j = m + n$ and $p, q \leqslant 0$, and hence

$$\left[N^{(+)(0)}((j); p, q)\right]^{r'(j)}_{r(j)} = \left(T^{(0)}_{(+)}, \left[\widetilde{T}_{n,m}\right]_{r(j)} \left[\widetilde{T}_{m,n}\right]^{r'(j)} T^{(0)}_{(+)}\right)_{\mathcal{W}^{(+)}} = \delta^{\{r'(j)\}}_{\{r(j)\}} N^{(+)(0)}_{m,n} \,, \tag{3.100}$$

$$\left[\widetilde{T}_{m,n}\right]_{r(j)} \equiv \left[(\widetilde{L}^+)^m(\widetilde{L}^-)^n T^{(0)}_{(+)}\right]_{\{r(j)\}} = \widetilde{L}^+_{\{r_1} \left[\widetilde{T}_{m-1,n}\right]_{r(j-1)\}} = \widetilde{L}^-_{\{r_1} \left[\widetilde{T}_{m,n-1}\right]_{r(j-1)\}} \,,$$

where the matrix elements

$$N^{(+)(0)}_{m,n} = \frac{1}{\dim(j)} \left(T^{(0)}_{(+)}, \left[\widetilde{T}_{n,m}\right]_{r(j)} \left[\widetilde{T}_{m,n}\right]^{r(j)} T^{(0)}_{(+)}\right)_{\mathcal{W}^{(+)}} \,, \tag{3.101}$$

with $\dim(j) = \delta^{\{r(j)\}}_{\{r(j)\}}$. These matrix elements obey the recursion relation

$$N^{(+)(0)}_{m,n} = \frac{\dim(j-1)}{\dim(j)} D^+_{m,n} N^{(+)(0)}_{m,n-1} = \frac{\dim(j-1)}{\dim(j)} D^-_{m,n} N^{(+)(0)}_{m-1,n} \,, \tag{3.102}$$

where the coefficients $D^{\pm}_{m,n}$ are defined by

$$\widetilde{L}^+_s \left[\widetilde{T}_{m,n}\right]_{sr(j-1)} = D^+_{m,n} \left[\widetilde{T}_{m,n-1}\right]_{r(j-1)} \,, \quad \widetilde{L}^-_s \left[\widetilde{T}_{m,n}\right]_{sr(j-1)} = D^-_{m,n} \left[\widetilde{T}_{m-1,n}\right]_{r(j-1)} \,,$$

and given by

$$D^+_{m,n} = D^-_{n,m} = \frac{2n(n + \epsilon_0 - 1)(m + n + 2\epsilon_0 - 1)}{m + n + \epsilon_0 - 1} \quad \text{for } m \geqslant 0 \text{ and } n \geqslant 1 \,, \tag{3.103}$$

as can be seen by using $(n = 0, 1, 2, \dots)$

$$\widetilde{L}^+_r \widetilde{L}^-_r \widetilde{T}_{0,n} = \mu_n \widetilde{T}_{0,n} 0 \,, \qquad \mu_n = 2(n + 1)(n + 2\epsilon_0) \,, \tag{3.104}$$

$$\widetilde{x}^+ \widetilde{T}_{0,n} = \mu'_n \widetilde{T}_{1,n-1} + \widetilde{T}' \,, \qquad \mu'_n = 4n(n + \epsilon_0 - 1) \,, \tag{3.105}$$

where the first formula follows from $(\widetilde{L}^+_r \widetilde{L}^-_r - 4\epsilon_0)T^{(0)}_{0;(0)} = 0$, and $\widetilde{T}'$ is a descendant of $\widetilde{x}^{\pm} T^{(0)}_{0;(0)}$ that decouples from $(\cdot, \cdot)_{\mathcal{W}^{(+)}}$. Since $D^+_{m,n} > 0$ for $m \geqslant 0$ and $n \geqslant 1$ it follows that $N^{(+)(0)}_{m,n} > 0$ for all $m, n \geqslant 0$ and hence $(\cdot, \cdot)_{\mathcal{W}^{(+)}}$ is positive definite in $\mathcal{W}^{(+)}_{(0)}$.

3.6 Expansion of Lorentz tensors in compact basis

3.6.1 Decomposition of the identity

Attempting to invert the harmonic map (3.2) from $\mathcal{T}$ to $\mathcal{M}$ amounts to seeking a decomposition of the unity $\mathbb{1}$ in the compact basis. From $[M_{ab},\mathbb{1}]_\star=0$, that is

$$[M_{rs},\mathbb{1}]_\star=0\,,\qquad [M_{0r},\mathbb{1}]_\star\ =\ (\widetilde{L}^+_r+\widetilde{L}^-_r)\mathbb{1}\ =\ 0\,,\tag{3.106}$$

and from the nature of the indecomposable structure (3.76) of $\mathcal{M}_{(0)}$, it follows that

$$\mathbb{1}_{\mathcal{M}}\ =\ (1+\pi)\sum_{\mathfrak{V}=\mathcal{W}^{(\pm)},\mathfrak{D}',\mathfrak{D}}\mathcal{N}_{\mathfrak{V}}\,\psi_{\mathfrak{V}}(\widetilde{x}^+)T^{(0)}_{e_0^{(\mathfrak{V})};(0)}\,,\qquad \psi_{\mathfrak{V}}(0)\ =\ 1\ ,\tag{3.107}$$

where $e_0^{(\mathfrak{D})}=2\epsilon_0$, $e_0^{(\mathfrak{D}')}=2$, $e_0^{(-)}=1$, $e_0^{(+)}=0$; $\mathcal{N}_{\mathfrak{V}}$ are normalization constants; and the embedding functions $\psi_{\mathfrak{V}}(\widetilde{x}^+)$ account for the *regular* contributions to $\mathbb{1}_{\mathcal{M}}$ from $\mathfrak{V}$ plus eventual *logarithmic* contributions from other modules that arise if $\mathfrak{V}$ contains a finite number of $(\mathfrak{e}\oplus\mathfrak{s})$-types with $\vec{s}=0$. Using (3.23), (3.24), (3.25) and (3.26), the condition (3.106) can be shown to imply that

$$D_{2\epsilon_0}\psi_{\mathfrak{D}}(x^+)=D_2\psi_{\mathfrak{D}'}(x^+)\ =\ 0\,,\qquad D_1\psi_{(-)}(x^+)=\frac{2\epsilon_0-1}{x^+}\,,\tag{3.108}$$

$$D_2\chi_{(+)}(x^+)=\frac{4\epsilon_0}{x^+}\,,\qquad \psi_{(+)}(x^+)=1-\frac{1}{2\epsilon_0}x^+\chi_{(+)}(x^+)\,,\tag{3.109}$$

where $D_e(x)=4x\frac{d^2}{dx^2}+4(e-\epsilon_0)\frac{d}{dx}+1+(e-2\epsilon_0)(e-2)x^{-1}$. The transformations

$$\psi_{\mathfrak{D}}(x^+)=y^{-\nu}J(y)\,,\qquad \psi_{\mathfrak{D}'}(x^+)\ =\ y^{-\nu'}J(y)\,,\tag{3.110}$$

$$\psi_{(-)}(x^+)=1-\sqrt{\pi}\left(\frac{y}{2}\right)^{1-\nu'}J(y)\,,\qquad \chi_{(+)}(x^+)\ =\ y^{-\nu'}J(y)\ ,\tag{3.111}$$

with $\nu=\epsilon_0-1$ and $\nu'=1-\epsilon_0$, lead to Bessel's differential equation[30]

$$B_\nu J\ =\ K_{\mathfrak{V}}\,,\qquad B_\nu\ =\ \frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}+1-\frac{\nu^2}{y^2}\,,\qquad \widetilde{x}^+\ =\ y^2\ ,\tag{3.112}$$

[30]The Bessel, Struve, Neumann and modified Struve functions, respectively, have the series expansions

$$J_\nu(y)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{y}{2}\right)^{\nu+2n}}{n!\Gamma(\nu+n+1)}\,,\qquad \mathbf{H}_\nu(y)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{y}{2}\right)^{\nu+1+2n}}{\Gamma(n+\frac{3}{2})\Gamma(\nu+n+\frac{3}{2})}\,,$$

$$\pi N_p(y)=\mathbf{f}_p(y)-\sum_{n=0}^{p-1}\frac{(p-n-1)!}{n!}\left(\frac{y}{2}\right)^{2n-p}\,,\quad \widetilde{\mathbf{H}}_{-p-\frac{1}{2}}(y)=\frac{1}{p!}\left(\mathbf{f}_{-p-\frac{1}{2}}(y)-\sum_{n=0}^{p-1}\frac{(p-1-n)!}{\Gamma(n+\frac{3}{2})}\left(\frac{y}{2}\right)^{2n+\frac{1}{2}-p}\right)\,,$$

where $p\in\{0,1,2,\dots\}$ and

$$\mathbf{f}_\nu(y)\ =\ \sum_{n=0}^{\infty}\frac{(-1)^n(\log y^2-\psi(n+1)-\psi(n+\nu+1))}{n!\Gamma(\nu+n+1)}\left(\frac{y}{2}\right)^{\nu+2n}\,,$$

where $\psi(z)=\frac{d}{dz}\log\Gamma(z)$. These functions obey

$$B_\nu J_\nu=B_pN_p\ =\ 0\,,\qquad B_\nu\mathbf{H}_\nu=\frac{1}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\left(\frac{y}{2}\right)^{\nu-1}\,,$$

$$B_{-p-\frac{1}{2}}\widetilde{\mathbf{H}}_{-p-\frac{1}{2}}=\frac{1}{\sqrt{\pi}}\left(\frac{y}{2}\right)^{-p-\frac{3}{2}}\,,\qquad B_\nu\mathbf{f}_\nu=\frac{1}{\Gamma(\nu)}\left(\frac{y}{2}\right)^{\nu-2}\,.$$

JHEP10(2008)022

where the sources are given by

$$K_{\mathfrak{D}}=K_{\mathfrak{D}'}\ =\ 0\,,\qquad K_{(-)}\ =\ \frac{1}{\sqrt{\pi}}\left(\frac{y}{2}\right)^{\nu'-1}\,,\qquad K_{(+)}\ =\ 4\epsilon_0 y^{\nu'-2}\,. \tag{3.113}$$

As a result, the regular embeddings are given by

$$D\geqslant 4: \qquad \psi_{\mathfrak{D}}\ =\ \Gamma(\epsilon_0)\left(\frac{y}{2}\right)^{1-\epsilon_0}J_{\epsilon_0-1}(y)\,, \tag{3.114}$$

$$D=4,6,\cdots: \qquad \psi_{\mathfrak{D}'}\ =\ \Gamma(2-\epsilon_0)\left(\frac{y}{2}\right)^{\epsilon_0-1}J_{1-\epsilon_0}(y)\,, \tag{3.115}$$

$$D=5,7,\cdots: \qquad \psi_{\mathcal{W}^{(-)}}\ =\ 1-\sqrt{\pi}\Gamma(\frac{3}{2}-\epsilon_0)\left(\frac{y}{2}\right)^{\epsilon_0}\mathbf{H}_{1-\epsilon_0}(y)\,, \tag{3.116}$$

where we note that $\mathfrak{D}=\mathcal{W}^{(-)}$ in $D=4$ and $\mathfrak{D}=\mathfrak{D}'$ in $D=5$. The irregular embeddings involving linear logarithms are given by

$$D=7,9,\cdots: \qquad \psi_{\mathfrak{D}'}\ =\ -\frac{\pi}{(\epsilon_0-2)!}\left(\frac{y}{2}\right)^{\epsilon_0-1}N_{\epsilon_0-1}(y), \tag{3.117}$$

$$D=6,8,\cdots: \qquad \psi_{\mathcal{W}^{(-)}}\ =\ 1-\sqrt{\pi}\left(\frac{y}{2}\right)^{\epsilon_0}\widetilde{\mathbf{H}}_{1-\epsilon_0}(y)\,, \tag{3.118}$$

$$D=4,6,\cdots: \qquad \psi_{\mathcal{W}^{(+)}}\ =\ 1-2\Gamma(1-\epsilon_0)\left(\frac{y}{2}\right)^{\epsilon_0+1}\mathbf{f}_{1-\epsilon_0}(y)\,. \tag{3.119}$$

If $D=5,7,\ldots$ the function $\psi_{(+)}$ contains quadratic logarithms and this case is left out in this work. We note that $[E,T^{(0)}_{e;(0)}]_\star=0$ and more generally that

$$[E,f(\widetilde{x}^+,\widetilde{x}^-)T^{(0)}_{e;(0)}]_\star=0 \text{ for differentiable } f\,, \tag{3.120}$$

which together with $[M_{ab},\mathbb{1}_{\mathcal{M}}]_\star=0$ implies that $[P_a,\mathbb{1}_{\mathcal{M}}]_\star=0$. Conversely, the general solution to $[M_{AB},\mathbf{M}]_\star=0$ is $\mathbf{M}=\sum_{\mathfrak{W}=\mathfrak{D}^+,\mathfrak{D}^-,\mathfrak{D}'^+,\mathfrak{D}'^-,\mathcal{W}^{(\pm)}}\mathcal{N}_{\mathfrak{W}}\mathbf{M}_{\mathfrak{W}}$, where $\mathcal{N}_{\mathfrak{W}}$ are constants and $\mathbf{M}_{\mathfrak{D}^\pm}=\psi_{\mathfrak{D}}(x^\pm)T^{(0)}_{\pm2\epsilon_0;(0)}$, $\mathbf{M}_{\mathfrak{D}'^\pm}=\psi_{\mathfrak{D}'}(x^\pm)T^{(0)}_{\pm2;(0)}$ and $\mathbf{M}_{\mathcal{W}^{(\pm)}}=(1+\pi)\psi_{\mathcal{W}^{(\pm)}}(x^+)T^{(0)}_{e_0^{(\pm)};(0)}$. Demanding $\pi(\mathbf{M})=\mathbf{M}$ enforces $\mathcal{N}_{\mathfrak{D}^+}=\mathcal{N}_{\mathfrak{D}^-}$ and $\mathcal{N}_{\mathfrak{D}'^+}=\mathcal{N}_{\mathfrak{D}'^-}$, resulting in (3.107). The final determination of $\mathcal{N}_{\mathfrak{V}}$ from the condition $\mathbb{1}_{\mathcal{M}}\star\mathbb{1}_{\mathcal{M}}=\mathbb{1}_{\mathcal{M}}$ requires a regularization of $\star$-product compositions of various special functions. We shall address this issue elsewhere [36], and continue here with the derivation of the embedding functions and the expansion of Lorentz tensors in compact basis.

3.6.2 Decomposition of Lorentz tensors

The Lorentz tensor $|S_{(s+k,s)}\rangle_{12}=S^{a(s+k),b(s)}|T_{a(s+k),b(s)}\rangle_{12}=S^{a(s+k),b(s)}T_{a(s+k),b(s)}(1)|\mathbb{1}\rangle_{12}$ can be expanded in the compact basis as follows

$$|S_{(s+k,s)}\rangle=\sum_{s+k\geqslant j_1\geqslant s\geqslant j_2\geqslant 0}S^{r(j_1),t(j_2)}_{(s+k,s)}\sum_{\mathfrak{V}=\mathfrak{D},\mathfrak{D}',\mathcal{W}^{(\pm)}}\mathcal{N}^{(s+k,s)}_{\mathfrak{V};(j_1,j_2)}|^{(s+k,s)}_{(j_1,j_2)}\rangle_{\mathfrak{V};r(j_1),t(j_2)}\,, \tag{3.121}$$

where:

(i) $S^{r(j_1),t(j_2)}_{(s+k,s)}=S^{0(s+k-j_1)\{r(j_1),t(j_2)\}0(s-j_2)}$ are the type-(j_1,j_2) *polarization tensors* contained in $S^{a(s+k),b(s)}$;

(ii) $\mathcal{N}^{(s+k,s)}_{\mathfrak{V};(j_1,j_2)}$ are *normalization constants*;

(iii) the *embedded polarizations*

$$|{}^{(s+k,s)}_{(j_1,j_2)}\rangle_{\mathfrak{V}} = (1+\pi)|{}^{(s+k,s)}_{(j_1,j_2)}\rangle^+_{\mathfrak{V}}, \qquad s+k \geqslant j_1 \geqslant s \geqslant j_2 \geqslant 0\,, \tag{3.122}$$

$$|{}^{(s+k,s)}_{(j_1,j_2)}\rangle^+_{\mathfrak{V}} = \psi^{(s+k,s)}_{\mathfrak{V};(j_1,j_2)}(x^+)|(s);e_0^{(\mathfrak{V})};(j_1,j_2)\rangle^+_{\mathfrak{V}}, \tag{3.123}$$

where $|(s);e_0^{(\mathfrak{V})};(j_1,j_2)\rangle^+_{\mathfrak{V}} = T^{(s)}_{e_0^{(\mathfrak{V})};(j_1,j_2)}(1)|\mathbb{1}\rangle_{12}$ and $e_0^{(\mathfrak{V})}$ is the minimal positive energy compatible with the embedding into $\mathfrak{V}$, which is given for $\mathfrak{V}=\mathfrak{D},\mathcal{W}^{(\pm)}$ by (see also (3.136))

$$e_0^{(\mathfrak{D})} = 2\epsilon_0+j_1+j_2\ , \qquad \begin{cases} e_0^{(\pm)} \in \{0,1\} \\ e_0^{(\pm)}+j_1+j_2 \ = \ \frac{1}{2}(1\mp 1) \end{cases} ; \tag{3.124}$$

(iv) the *embedding functions* $\psi^{(s+k,s)}_{\mathfrak{V};(j_1,j_2)}(x^+)$ are determined by $\psi^{(s+k,s)}_{\mathfrak{V};(j_1,j_2)}(0)=1$ and the requirement that the set $\left\{|{}^{(s+k,s)}_{(j_1,j_2)}\rangle_{\mathfrak{V}}\right\}_{s+k\geqslant j_1\geqslant s\geqslant j_2\geqslant 0}$ furnishes a type-$(s+k,s)$ Lorentz tensor, i.e.

$$\begin{aligned} M_{0r}|{}^{(s+k,s)}_{(j_1,j_2)}\rangle_{\mathfrak{V}} = {} & M^{(s+k,s)}_{(j_1,j_2);(1,0)}|{}^{(s+k,s)}_{(j_1+1,j_2)}\rangle_{\mathfrak{V}} + M^{(s+k,s)}_{(j_1,j_2);(-1,0)}|{}^{(s+k,s)}_{(j_1-1,j_2)}\rangle_{\mathfrak{V}} \\ & + M^{(s+k,s)}_{(j_1,j_2);(0,1)}|{}^{(s+k,s)}_{(j_1,j_2+1)}\rangle_{\mathfrak{V}} + M^{(s+k,s)}_{(j_1,j_2);(0,-1)}|{}^{(s+k,s)}_{(j_1,j_2-1)}\rangle_{\mathfrak{V}}\ , \end{aligned} \tag{3.125}$$

for $M_{0r}=\frac{1}{2}(L_r^+ + L_r^-)$ and $M^{(s+k,s)}_{(s+k,j_2);(1,0)} = M^{(s+k,s)}_{(j_1,s);(0,1)} = 0$, and $M^{(s+k,s)}_{(s,j_2);(-1,0)}=0$ for $j_2<s$. In $\mathcal{M}$ the condition $j_1\geqslant s\geqslant j_2$ is obeyed identically, and (3.125) is equivalent to

$$M_{0\{r}|{}^{(s+k,s)}_{(s+k,0)}\rangle_{\mathfrak{V};r(s+k)\}} = 0\ , \tag{3.126}$$

providing a differential equation for $\psi^{(s+k,s)}_{\mathfrak{V};(s+k,0)}(x^+)$.

Since $|S_{(0,0)}\rangle_{12} = S_{(0,0)}|\mathbb{1}\rangle_{12}$ where $S_{(0,0)}$ is a constant, it follows that (3.121) is equivalent to (3.107) for $s=k=0$. In this case (3.125) reads $(M_{ab}(1)+M_{ab}(2))|\mathbf{M}_{\mathfrak{V}}\rangle_{12}=0$, where the *reduced reflector*

$$|\mathbf{M}_{\mathfrak{V}}\rangle_{12} = |{}^{(0,0)}_{(0,0)}\rangle_{\mathfrak{V}} \ = \ (1+\pi)\psi^{(0,0)}_{\mathfrak{V};(0,0)}(x^+)T^{(0)}_{e_0^{(\mathfrak{V})};(0)}|\mathbb{1}\rangle_{12}\ . \tag{3.127}$$

Eq. (3.120) can be written as $[E(1)-E(2),f(x^+)]_\star=0$ where $x^+=2L_r^+(1)L_r^+(2)$, which implies that $(E(1)-E(2))|\mathbf{M}_{\mathfrak{V}}\rangle_{12}=0$. Thus $(P_a(1)-P_a(2))|\mathbf{M}_{\mathfrak{V}}\rangle_{12}=0$, and hence one may identify

$$|\mathbb{1}\rangle_{12} = \sum_{\mathfrak{V}}\mathcal{N}^{(0,0)}_{\mathfrak{V};(0,0)}|\mathbf{M}_{\mathfrak{V}}\rangle_{12}\,, \qquad \mathcal{N}_{\mathfrak{V}} \ = \ \mathcal{N}^{(0,0)}_{\mathfrak{V};(0,0)}\,, \quad \psi_{\mathfrak{V}} \ = \ \psi^{(0,0)}_{\mathfrak{V};(0,0)}\ . \tag{3.128}$$

Using (A.5) and (A.6) the overlap conditions can also be written as

$$P_r(1)|\mathbf{M}_{\mathfrak{V}}\rangle_{12} = \frac{i}{2}L_r^+|\mathbf{M}_{\mathfrak{V}}\rangle_{12} \ = \ -\frac{i}{2}L_r^-|\mathbf{M}_{\mathfrak{V}}\rangle_{12}\ , \tag{3.129}$$

$$M_{r0}(1)|\mathbf{M}_{\mathfrak{V}}\rangle_{12} = -\frac{1}{2}(L_r^+(1)-L_r^+(2))|\mathbf{M}_{\mathfrak{V}}\rangle_{12}\ , \tag{3.130}$$

$$M_{rs}(1)|\mathbf{M}_{\mathfrak{V}}\rangle_{12} \;=\; (L^+_{[r}(1)L^+_{s]}(2) + L^-_{[r}(1)L^-_{s]}(2))|\mathbf{M}_{\mathfrak{V}}\rangle_{12} \;, \tag{3.131}$$

$$E(1)|\mathbf{M}_{\mathfrak{V}}\rangle_{12} \;=\; \frac{1}{2(D-1)}(L^+_r(1)L^+_r(2) - L^-_r(1)L^-_r(2))|\mathbf{M}_{\mathfrak{V}}\rangle_{12} \;. \tag{3.132}$$

It follows that $(P_{\{r_1}\cdots P_{r_k\}})(1)|\mathbf{M}_{\mathfrak{V}}\rangle_{12} = (2i)^{-k}(-1)^l L^{+(l)}_{\{r(l)}L^{-(k-l)}_{r(k-l)\}}|\mathbf{M}_{\mathfrak{V}}\rangle_{12}$ for $l = 0,1,\ldots,k$. Combined with (3.127) and $(P_{\{r_1}\cdots P_{r_k\}})(1)|\mathbf{M}_{\mathfrak{V}}\rangle_{12} = (1+\pi)\psi^{(k,0)}_{\mathfrak{V};(k,0)}(x^+)|(0);e^{(\mathfrak{V})}_0;(k,0)\rangle_{\mathfrak{V};r(k)}$ where $|(0);e^{(\mathfrak{V})}_0;(k,0)\rangle_{\mathfrak{V};r(k)} = f^{(\mathfrak{V})}_{r(k)}|(0);e^{(\mathfrak{V})}_0;(0)\rangle$ with $f^{(\mathfrak{V})}_{r(k)}$ being a monomial of degree k in L^+_r and (L^+_r, L^-_r) for $\mathfrak{V} = \mathfrak{D},\mathfrak{D}'$ and $\mathfrak{V} = \mathcal{M}^{(\pm)}$, respectively, it follows that $(k = 0,1,2,\ldots)$

$$\psi^{(k,0)}_{\mathfrak{V};(k,0)}(x^+) = \psi^{(0,0)}_{\mathfrak{V};(0,0)}(x^+) \;=\; \psi_{\mathfrak{V}}(x^+) \qquad \text{for } \mathfrak{V} = \mathfrak{D},\mathfrak{D}' \;. \tag{3.133}$$

The energy operator, on the other hand, acts as

$$Ef(x^+)|(s);e;(j_1,j_2)\rangle \;=\; \left(2x^+\frac{d}{dx^+} + e\right) f(x^+)|e;(j_1,j_2)\rangle \;,$$

which means that the functional forms of $\psi^{(k,0)}_{\mathfrak{V};(j_1,0)}(x^+)$ with $j_1 < k$ differ from that of $\psi_{\mathfrak{V}}(x^+)$ for $\mathfrak{V} = \mathfrak{D},\mathfrak{D}'$. For example, from $|^{(k,0)}_{\mathfrak{D};(k-1,0)}\rangle \propto E|^{(k,0)}_{\mathfrak{D};(k,0)}\rangle$ it follows that $\psi^{(k,0)}_{\mathfrak{D};(k-1,0)}(x^+) = \frac{1}{2\epsilon_0+k}(2x^+\frac{d}{dx^+} + 2\epsilon_0 + k)\psi^{(0,0)}_{\mathfrak{D};(0,0)}(x^+)$.

Next we turn to closer look at the embedding functions in the different sectors.

3.6.3 Composite-massless sector

A state $L^+_{r_1}\cdots L^+_{r_l}|2\epsilon_0+s;(s)\rangle_{t(s)} \in \mathfrak{D}(2\epsilon_0+s;(s))$ at excitation level l can be decomposed under $\mathfrak{s}$ using $L^+_r|2\epsilon_0+s;(s)\rangle_{rt(s-1)} = 0$ with the result

$$\begin{aligned} L^+_{r_1}\cdots L^+_{r_l}|2\epsilon_0+s;(s)\rangle_{t(s)} \;=&\; \sum_{n=0}^{[l/2]}\;\sum_{p=0}^{\min(s,l-2n)} (x^+)^n \\ &\times|(s);2\epsilon_0+l-2n;(s+l-2n-p,p)\rangle \;. \end{aligned} \tag{3.134}$$

Thus, a state $|(s);(j_1,j_2)\rangle \in \mathfrak{D}(s+2\epsilon_0;(s))$ of type-(j_1,j_2) is of the form

$$|(s);(j_1,j_2)\rangle \;=\; \sum_{n=0}^{\infty}\psi_{(j_1,j_2);n}(x^+)^n|(s);e^{(\mathfrak{D})}_0;(j_1,j_2)\rangle \;, \tag{3.135}$$

where $\psi_{(j_1,j_2);n}$ are arbitrary coefficients,

$$|(s);e^{(\mathfrak{D})}_0;(j_1,j_2)\rangle_{r(j_1),t(j_2)} = L^{+(j_1-s)}_{\{r(j_1-s)}L^{+(j_2)}_{t(j_2)}|2\epsilon_0+s;(s)\rangle_{r(s)\}} \;, \quad e^{(\mathfrak{D})}_0 = j_1+j_2+2\epsilon_0 \;, \tag{3.136}$$

is the type-(j_1,j_2) state in $\mathfrak{D}(2\epsilon_0+s;(s))$ of minimal energy, and the series expansion does not collapse for any values of s,k,j_1 and j_2 since $(x^+)^n|(s);e^{(\mathfrak{D})}_0;(j_1,j_2)\rangle$ are not null states. The lemma (2.40) implies

$$M_{0\{r}(x^+)^n|2\epsilon_0+p;(p)\rangle_{r(p)\}} = \frac{1}{2}(1+4n(n+\epsilon_0-1))L^+_{\{r}(x^+)^{n-1}|2\epsilon_0+p;(p)\rangle_{r(p)\}} \;, \tag{3.137}$$

where $p = s + k$ and $|2\epsilon_0 + p; (p)\rangle_{r(p)} \equiv L^{+(k)}_{\{r(k)}|s + 2\epsilon_0; (s)\rangle_{r(s)\}}$ has the property $L^-_{\{r_1}|2\epsilon_0 + p; (p)\rangle_{r(p)\}} = 0$. Hence, independently of $s + k$, the embedding condition (3.126) implies

$$\left(4x^+\frac{d^2}{d(x^+)^2} + 4\epsilon_0\frac{d}{dx^+} + 1\right)\psi^{(s+k,s)}_{\mathfrak{D};(s+k,0)}(x^+) = 0\ . \tag{3.138}$$

The transformation $\psi^{(s+k,s)}_{\mathfrak{D};(s+k,0)}(x^+) = y^{-\nu}J(y)$ with $\nu = \epsilon_0 - 1$ and $x^+ = y^2$ brings (3.138) to Bessel's differential equation (3.112) with index ν and source $K = 0$. Thus

$$\psi^{(s+k,s)}_{\mathfrak{D};(s+k,0)}(x^+) = \psi^{(0,0)}_{\mathfrak{D};(0,0)}(x^+) \ = \ \psi_{\mathfrak{D}}(x^+) \ = \ \Gamma(\epsilon_0)\left(\frac{y}{2}\right)^{1-\epsilon_0} J_{\epsilon_0-1}(y)\,, \tag{3.139}$$

as given also in (3.114). For even D the embedding functions are algebraic powers times trigonometric functions. In particular,

$$D = 4 : |\mathbf{M}_{\mathfrak{D}}\rangle_{12} \ = \ (1 + \pi)\cos(y)|1; (0)\rangle_{12}\ . \tag{3.140}$$

Alternatively, the Flato-Fronsdal formula (3.64) yields the following expression for the contribution from the $\mathfrak{D}^+$ sector to the right-hand side of the embedding formula (3.121):

$$\begin{aligned}\sum_{s+k\geqslant j_1\geqslant s\geqslant j_2\geqslant 0} & S^{r(j_1),t(j_2)}_{(s+k,s)}\mathcal{N}^{(s+k,s)}_{\mathfrak{D};(j_1,j_2)}\psi^{(s+k,s)}_{\mathfrak{D};(j_1,j_2)}(x^+)f^{(j_1,j_2)}_{(s)}\\ & \times L^{+(j_1-s)}_{\{r(j_1-s)}L^{+(j_2)}_{t(j_2)}f_{r(s)\}}(1,2)|2\epsilon_0; (0)\rangle_{12}\ ,\end{aligned} \tag{3.141}$$

where $f^{(j_1,j_2)}_{(s)}$ are normalizations and $L^+_r = L^+_r(1) + L^+_r(2)$. The left-hand side of (3.121) decomposes under $\mathfrak{s}$ as

$$\sum_{s+k\geqslant j_1\geqslant s\geqslant j_2\geqslant 0} g^{(s+k,s)}_{(j_1,j_2)}S^{r(j_1),t(j_2)}_{(s+k,s)}T_{0(s+k-j_1)\{r(j_1),t(j_2)\}0(s-j_2)}(1)|\mathbb{1}\rangle_{12}\,, \tag{3.142}$$

where $g^{(s+k,s)}_{(j_1,j_2)}$ are embedding coefficients and $T_{0(s+k-j_1)\{r(j_1),t(j_2)\}0(s-j_2)} = (M_{rs})^{\star j_2}\star (M_{r0})^{\star(s-j_2)}\star(P_r)^{\star(j_1-s)}\star E^{\star(k-j_1+s)}$ plus trace corrections. (Anti)-symmetrizing under $1\leftrightarrow 2$ using eqs. (3.129)–(3.132), and rewriting the $\mathfrak{D}^+$ sector using (3.127) and

$$L^-_{[r}(1)L^-_{s]}(2)\psi_{\mathfrak{D}}(x^+)|2\epsilon_0; (0)\rangle_{12} = L^+_{[r}(1)L^+_{s]}(2)(D_M\psi_{\mathfrak{D}})(x^+)|2\epsilon_0; (0)\rangle_{12})\,, \tag{3.143}$$

$$L^-_r(1)L^-_r(2)\psi_{\mathfrak{D}}(x^+)|2\epsilon_0; (0)\rangle_{12} = (D_E\psi_{\mathfrak{D}})(x^+)|2\epsilon_0; (0)\rangle_{12})\,, \tag{3.144}$$

where D_M and D_E are differential operators in x^+ with coefficients that are analytic at $x^+ = 0$, the type-(j_1, j_2) contribution to the $\mathfrak{D}^+$ sector of (3.142) takes the form

$$S^{r(j_1),t(j_2)}_{(s+k,s)}\mathcal{N}_{\mathfrak{D}}(D^{(s+k,s)}_{(j_1,j_2)}\psi_{\mathfrak{D}})(x^+)M_{r(j_1),t(j_2)}(1,2)|2\epsilon_0; (0)\rangle_{12}\,, \tag{3.145}$$

where $D^{(s+k,s)}_{(j_1,j_2)}$ are analytic differential operators in x^+ (including the coefficients $g^{(s+k,s)}_{(j_1,j_2)}$) and $M_{r(j_1),t(j_2)}(1,2) = (-1)^s M_{r(j_1),t(j_2)}(2,1)$ is a normalized monomial in $L^+_r(1)$ and $L^+_r(2)$ of degree $2j_2+s-j_2+j_1-s = j_1+j_2$. Equating (3.141) and (3.145) yields $M_{r(j_1),t(j_2)}(1,2) = L^{+(j_1-s)}_{\{r(j_1-s)}L^{+(j_2)}_{t(j_2)}f_{r(s)\}}(1,2)$, and hence

$$\psi^{(s+k,s)}_{\mathfrak{D};(j_1,j_2)}(x^+) \ = \ \frac{\mathcal{N}_{\mathfrak{D}}}{f^{(j_1,j_2)}_{(s)}\mathcal{N}^{(s+k,s)}_{\mathfrak{D};(j_1,j_2)}}\left(D^{(s+k,s)}_{(j_1,j_2)}\psi_{\mathfrak{D}}\right)(x^+)\ .$$

In particular, from $M_{r0}=-\frac{1}{2}(L^+_r+L^-_r)$, $P_r=\frac{i}{2}(L^+_r-L^-_r)$ and $(L^\pm_r(1)+L^\mp_r(2))|\mathbb{1}\rangle_{12}=0$ it follows that the type-$(s+k,0)$ contribution to the $\mathfrak{D}^+$ sector of (3.142) is proportional to $S^{r(s+k)}_{(s+k,0)}(L^+_r(1)+L^+_r(2))^k(L^+_r(1)-L^+_r(2))^s\psi^{(0,0)}_{\mathfrak{D};(0,0)}(x^+)|2\epsilon_0;(0)\rangle$, which contains no factors of M_{rs} and E, and hence

$$\psi^{(s+k,s)}_{\mathfrak{D};(s+k,0)}(x^+)=\psi_{\mathfrak{D}}(x^+)\,,\qquad D^{(s+k,s)}_{(s+k,s)}\;=\;\frac{f^{(j_1,j_2)}_{(s)}\mathcal{N}^{(s+k,s)}_{\mathfrak{D};(j_1,j_2)}}{\mathcal{N}_{\mathfrak{D}}}\,. \tag{3.146}$$

The trace corrections turns the binomial expansion of $(L^+_r(1)-L^+_r(2))^s$ into the Flato-Fronsdal formula (3.64). For example, the embedding into $\mathfrak{D}^+(2\epsilon_0+2;(2))$ of a spin-2 Weyl tensor $S^{a(2),b(2)}T_{a(2),b(2)}$, which decomposes under $\mathfrak{so}(D-1)$ into $(j_1,j_2)\in\{(2,0),(2,1),(2,2)\}$, contains the type-$(2,0)$ contribution $S^{r(2)}T_{r(2),0(2)}(1)=\frac{4}{3}S^{r(2)}(M_{r_10}\star M_{r_20}+\frac{1}{2\epsilon_0+1}P_{r_1}\star P_{r_2})$ as can be seen using (2.24) and (B.5) to expand

$$M_{\{a(2),b(2)\}_D}=\frac{4}{3}M_{a_1b_1}\star M_{a_2b_2}-\frac{4}{3(2\epsilon_0+1)}\big(\eta_{a(2)}P_{b_1}\star P_{b_2}-2\eta_{a_1b_1}P_{(a_2}\star P_{b_2)}+\eta_{b(2)}P_{a_1}\star P_{a_2}\big)$$
$$+\frac{4\epsilon_0}{3(2\epsilon_0+1)}(\eta_{a(2)}\eta_{b(2)}-\eta_{a_1b_1}\eta_{a_2b_2})\,,$$

and make contact with (3.64): $S^{r(2)}T_{r(2),0(2)}(1)\psi_{\mathfrak{D}}(x^+)|2\epsilon_0;(0)\rangle=$

$$=\frac{2\epsilon_0}{3(2\epsilon_0+1)}S^{r(2)}\psi_{\mathfrak{D}}(x^+)\left(L^+_{r_1}(1)L^+_{r_2}(1)-\frac{2(\epsilon_0+1)}{\epsilon_0}L^+_{r_1}(1)L^+_{r_2}(2)+L^+_{r_1}(2)L^+_{r_2}(2)\right)|2\epsilon_0;(0)\rangle_{12}$$
$$=\frac{2\epsilon_0}{3(2\epsilon_0+1)}S^{r(2)}\psi_{\mathfrak{D}}(x^+)|2\epsilon_0+2;(2,0)\rangle_{12;r_1r_2}\,.$$

3.6.4 Singleton factorization of composite-massless reflector

The form (3.139) of the embedding function $\psi_{\mathfrak{D}}=\psi^{(0,0)}_{\mathfrak{D};(0,0)}$ can also be derived starting from the compositeness relation (3.52), that is, the isomorphism $|2\epsilon_0;(0)\rangle_{12}\simeq|\epsilon_0;(0)\rangle_1\otimes|\epsilon_0;(0)\rangle_2$. The reduced reflector $|\mathbf{M}_{\mathfrak{D}}\rangle_{12}$ can thus be identified as the scalar-singleton reflector

$$|\mathbf{M}_{\mathfrak{D}}\rangle_{12}=\mathcal{N}_{\mathfrak{D}_0}(1+\pi)|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12}\,,\qquad|\mathbb{1}_{\mathfrak{D}_0}\rangle_{12}\;=\;R_2^{-1}(\mathbb{1}_{\mathfrak{D}_0})_{12}\,, \tag{3.147}$$

where $\mathcal{N}_{\mathfrak{D}_0}$ is a normalization constant, $\mathbb{1}_{\mathfrak{D}_0}$ is the identity operator in $\mathfrak{D}_0$, and the reflection map R induced from (2.48) and (2.49) is given by

$$R(X|\epsilon_0;(0)\rangle^\pm)\;=\;{}^\pm\langle\epsilon_0;(0)|(\tau\circ\pi)(X)\,,\quad R(X\star Y)\;=\;R(Y)\star R(X)\,, \tag{3.148}$$

for $X,Y\in\mathcal{A}$. It follows that $R(L^\pm_r)=-L^\mp_r$, $R(E)=E$, $R(M_{rs})=M_{rs}$ and $R(|n\rangle^\pm)=(-1)^n\,{}^\pm\langle n|$, where we have defined the following basis elements for $\mathfrak{D}_0$:

$$|n\rangle^\pm_{r(n)}=|\pm(n+\epsilon_0);(n)\rangle^\pm_{r(n)}\;=\;L^\pm_{r_1}\cdots L^\pm_{r_n}|\pm\epsilon_0;(0)\rangle^\pm\,, \tag{3.149}$$
$${}^\pm\langle n|_{r(n)}={}^\pm\langle\pm(n+\epsilon_0);(n)|_{r(n)}\;=\;{}^\pm\langle\epsilon_0;(0)|L^\mp_{r_1}\cdots L^\mp_{r_n}\,. \tag{3.150}$$

They are traceless as a consequence of the singular vector (A.15) and normalized to

$${}^\pm\langle m|^{r(m)}|n\rangle^\pm_{s(n)}=\delta_{mn}\mathcal{N}_n\delta^{\{r(n)\}}_{\{s(n)\}}\,,\qquad\mathcal{N}_n\;=\;2^n\,n!(\epsilon_0)_n\,, \tag{3.151}$$

as can be seen using $L_r^-|n\rangle_{s(n)} = 2n(n+\epsilon_0-1)\delta_{r\{s_1}|n-1\rangle_{s(n-1)\}}$. Defining the normal order ${}^\times_\times L_r^+ L_s^- {}^\times_\times = L_r^+ L_s^-$ and $z = 2L_r^+ L_r^-$, the unity $1\!\!1_{\mathfrak{D}_0} = \sum_{n=0}^\infty [\mathcal{N}_n]^{-1}|n\rangle\langle n|$ in $\mathfrak{D}_0$ assumes the form

$$1\!\!1_{\mathfrak{D}_0} = \sum_{n=0}^\infty {}^\times_\times \frac{(L_r^+ L_r^-)^n}{2^n n!(\epsilon_0)_n}|\epsilon_0;(0)\rangle\langle\epsilon_0;(0)|{}^\times_\times \;=\; \frac{\Gamma(\nu+1)2^\nu}{z^{\nu/2}}{}^\times_\times I_\nu(\sqrt{z})|\epsilon_0;(0)\rangle\langle\epsilon_0;(0)|{}^\times_\times\,, \quad (3.152)$$

where $\nu = \epsilon_0 - 1$ and the modified Bessel function $I_\nu(w) = e^{-\frac{i\pi\nu}{2}}J_\nu(e^{\frac{i\pi}{2}}w)$ for $-\pi < \arg w \leqslant \frac{\pi}{2}$ and $I_\nu(w) = e^{\frac{2i\pi\nu}{3}}J_\nu(e^{-\frac{3i\pi}{2}}w)$ for $-\frac{\pi}{2} < \arg w \leqslant \pi$. Applying $R^{-1} : \mathfrak{D}_0^* \mapsto \mathfrak{D}_0$ to the $\langle n|$ states in the expansion of $1\!\!1_{\mathfrak{D}_0}$ yields $(-1)^n|n\rangle$. Using $R_2^{-1}(z) = -2L_r^+(1)L_r^+(2) = -x$, which formally implies $R^{-1}(\sqrt{z}) = iy$, one finds

$$|1\!\!1_{\mathfrak{D}_0}\rangle_{12} = \frac{\Gamma(\nu+1)2^\nu}{(iy)^\nu}I_\nu(iy)|\epsilon_0;(0)\rangle_1|\epsilon_0;(0)\rangle_2 \;=\; \psi_{\mathfrak{D}}(x^+)|\epsilon_0;(0)\rangle_1|\epsilon_0;(0)\rangle_2\,, \quad (3.153)$$

with the embedding function given in (3.139) and in agreement with (3.147). In particular, in $D=4$ one has

$$1\!\!1_{\mathfrak{D}_0} = {}^\times_\times \cosh\sqrt{z}|\tfrac{1}{2};(0)\rangle\langle\tfrac{1}{2};(0)|{}^\times_\times\,, \qquad |1\!\!1_{\mathfrak{D}_0}\rangle_{12} \;=\; \cos y|\tfrac{1}{2};(0)\rangle_1|\tfrac{1}{2};(0)\rangle_2\,. \quad (3.154)$$

3.6.5 Remaining sectors and logarithmic contributions

In the sector $\mathfrak{D}'$ the embedding condition (3.106), or equivalently (3.126), implies

$$D_2\psi_{\mathfrak{D}'}(x^+) = \left(4x^+\frac{d^2}{d(x^+)^2} + 4(2-\epsilon_0)\frac{d}{dx^+} + 1\right)\psi_{\mathfrak{D}'}(x^+) \;=\; 0\,, \quad (3.155)$$

which upon $\psi_{\mathfrak{D}'}(x^+) = y^{-\nu'}J(y)$ with $\nu' = 1-\epsilon_0$ is transformed to Bessel's differential equation (3.112) with index ν' and source $K=0$. Taking into account also the boundary condition $\psi_{\mathfrak{D}'}(0) = 1$, it follows that $\psi_{\mathfrak{D}'}(x^+)$ is given by the rescaled Bessel functions in (3.115) if $\nu' \neq -1,-2,\dots$ and the rescaled Neumann functions in (3.117) if $\nu' = -1,-2,\dots$. In the latter case the lowest-weight space $\mathfrak{D}(2;(0))$ is the scalar p-lineton in $D = 5+2p = 7,9,\dots$, $p = 1,2,\dots$, and the logarithmic tail in $\psi_{\mathfrak{D}'}(x^+)$ starting at order $(x^+)^p\log x^+$ reflects the singular nature of the states $(x^+)^p|2;(0)\rangle$. In $D=4$ the isomorphism $|2;(0)\rangle_{12} \simeq |1;(\frac{1}{2})\rangle_1^i \otimes |1;(\frac{1}{2})\rangle_{2,i}$ implies that the reduced reflector $|\mathbf{M}_{\mathfrak{D}'}\rangle_{12}$ is proportional to the spinor-singleton reflector

$$D=4:\ |\mathbf{M}_{\mathfrak{D}'}\rangle_{12} \;=\; \mathcal{N}_{\mathfrak{D}_{1/2}}(1+\pi)|1\!\!1_{\mathfrak{D}_{\frac{1}{2}}}\rangle_{12}\,, \qquad |1\!\!1_{\mathfrak{D}_{1/2}}\rangle_{12} \;=\; R_2^{-1}1\!\!1_{\mathfrak{D}_{1/2}}, \quad (3.156)$$

where $\mathcal{N}_{\mathfrak{D}_{1/2}}$ is a normalization constant, the spinor-singleton identity operator

$$1\!\!1_{\mathfrak{D}_{1/2}} = \sum_{n=0}^\infty \left[\mathcal{N}_{n+\frac{1}{2}}\right]^{-1}|n+\tfrac{1}{2}\rangle^{i,r(n)}\langle n+\tfrac{1}{2}|_{i,r(n)}\,, \quad (3.157)$$

with γ-traceless basis states and normalizations given by

$$|n+\tfrac{1}{2}\rangle^\pm_{i,r(n)} = L^\pm_{r_1}\cdots L^\pm_{r_n}|\pm 1;\left(\tfrac{1}{2}\right)\rangle^\pm_i\,, \quad (3.158)$$

$${}^\pm\langle n+\tfrac{1}{2}|^{i,r(n)}|m+\tfrac{1}{2}\rangle^\pm_{j,s(m)} = \delta_{mn}\delta^i_j\delta^{\{r(n)\}}_{\{s(n)\}}\,, \quad \mathcal{N}_{n+\frac{1}{2}} \;=\; 2^n n!\left(\tfrac{3}{2}\right)_n\,, \quad (3.159)$$

as can be seen using $L_r^-|n+\frac{1}{2}\rangle_{i,s(n)} = 2n(n+\frac{1}{2})\delta_{r\{s_1}|n-\frac{1}{2}\rangle_{i,s(n-1)\}}$ where $\{\cdots\}$ denotes the γ-traceless projection, and the action of the reflection map is defined by

$$R(|\pm 1; \left(\frac{1}{2}\right)\rangle_i^{\pm}) = i^{\pm}\langle \pm 1; \left(\frac{1}{2}\right)|_i \,. \tag{3.160}$$

One has the following agreement:

$$D=4 : |\mathbb{1}_{\mathfrak{D}_{1/2}}\rangle_{12} \;=\; i\frac{\sin y}{y}|1;\left(\frac{1}{2}\right)\rangle_1^i\; |1;\left(\frac{1}{2}\right)\rangle_{2,i} \quad\leftrightarrow\quad \psi^{(0,0)}_{\mathfrak{D}';(0,0)}(x^+) = \frac{\sin y}{y}\,. \tag{3.161}$$

In the sector $\mathcal{W}^{(-)}$ the condition (3.106) reads

$$(\widetilde{L}_r^+ + \widetilde{L}_r^-)(1+\pi)\psi_{(-)}(\widetilde{x}^+)T^{(0)}_{1;(0)} \;=\; (1+\pi)\widetilde{L}_r^+(D_1\psi_{(-)})(\widetilde{x}^+)T^{(0)}_{1;(0)} \;=\; 0\,, \tag{3.162}$$

which is obeyed if

$$D_1\psi_{(-)}(x^+) = \left(4x^+\frac{d^2}{d(x^+)^2} + 4(1-\epsilon_0)\frac{d}{dx^+} + 1 + \frac{2\epsilon_0-1}{x^+}\right)\psi_{(-)}(x^+) = \frac{2\epsilon_0-1}{x_+}\,, \tag{3.163}$$

since (3.23) and (3.24) implies that

$$(1+\pi)\widetilde{L}_r^+\frac{1}{\widetilde{x}^+}T^{(0)}_{1;(0)} \;=\mathcal{C}_{-1}(1+\pi)\widetilde{L}_r^+T^{(0)}_{-1;(0)} = \mathcal{C}_{-1}(1+\mathcal{C}_1')\widetilde{L}_r^+T^{(0)}_{-1;(0)} = 0\,. \tag{3.164}$$

The transformation $\psi_{(-)}(x^+) = 1 + y^{\epsilon_0}J(y)$, $x^+ = y^2$, brings (3.163) to Bessel's differential equation (3.112) with index $\nu' = 1-\epsilon_0$ and source $K = -y^{-\epsilon_0} \;=\; -y^{\nu'-1}$, whose particular solution is the Struve function if $\nu' \neq -\frac{1}{2}, -\frac{3}{2}, \ldots$ and the modified Struve function if $\nu' = -\frac{1}{2}, -\frac{3}{2}, \ldots$, as given in (3.116) and (3.118). In the latter case $D = 4+2p = 6, 8, \ldots$, $p = 1, 2, \ldots$, and the logarithmic tail in $\psi_{(-)}(x^+)$ starting at order $(x^+)^p \log x^+$ corresponds to the fact that the states $(x^+)^p|1;(0)\rangle \in \mathfrak{D}(2\epsilon_0;(0)) \subset \mathfrak{D}$.

Finally, in the sector $\mathcal{W}^{(+)}$ the condition (3.106) reads

$$(\widetilde{L}_r^+ + \widetilde{L}_r^-)(1+\pi)\psi_{(+)}(\widetilde{x}^+)T^{(0)}_{0;(0)} \;=\; (1+\pi)\widetilde{L}_r^+(D_0\psi_{(-)})(\widetilde{x}^+)T^{(0)}_{0;(0)} \;=\; 0\,, \tag{3.165}$$

which is obeyed if

$$D_0\psi_{(-)}(x^+) \;=\; \left(4x^+\frac{d^2}{d(x^+)^2} - 4\epsilon_0\frac{d}{dx^+} + 1 + \frac{4\epsilon_0}{x^+}\right)\psi_{(-)}(x^+) \;=\; \frac{4\epsilon_0}{x_+} - 1\,, \tag{3.166}$$

since (3.23) and (3.24) implies that

$$(1+\pi)\widetilde{L}_r^+\left(\frac{4\epsilon_0}{\widetilde{x}^+}-1\right)T^{(0)}_{0;(0)} = \lim_{e\to 0}\left(\frac{4\epsilon_0}{\widetilde{x}^+}\widetilde{L}_r^+ + \frac{4\epsilon_0}{\widetilde{x}^-}\widetilde{L}_r^-\right)T^{(0}_{e;(0)} - (\widetilde{L}_r^+ + \widetilde{L}_r^-)T^{(0)}_{0;(0)} = 0\,. \tag{3.167}$$

The limiting procedure can be avoided by writing $\psi_{(+)}(x^+) = 1 - \frac{1}{2\epsilon_0}x^+\chi_{(+)}(x^+)$, that is

$$(1+\pi)\psi_{(+)}(\widetilde{x}^+)T^{(0)}_{0;(0)} \;=\; (1+\pi)\left(T^{(0)}_{0;(0)} + 2\chi_{(+)}(\widetilde{x}^+)T^{(0)}_{2;(0)}\right)\,. \tag{3.168}$$

The condition (3.106) then reads

$$(\widetilde{L}_r^+ + \widetilde{L}_r^-)(1+\pi)\left(T^{(0)}_{0;(0)} + 2\chi_{(+)}(\widetilde{x}^+)T^{(0)}_{2;(0)}\right) = 2(1+\pi)\widetilde{L}_r^+\left(T^{(0)}_{0;(0)} + (D_2\chi_{(+)})(\widetilde{x}^+)T^{(0)}_{2;(0)}\right) = 0\,,$$

which is obeyed if

$$D_2\chi_{(+)}(x^+) = \left(4x^+\frac{d^2}{d(x^+)^2} + 4(2-\epsilon_0)\frac{d}{dx^+} + 1\right)\psi_{(-)}(x^+) \ = \ \frac{4\epsilon_0}{x_+}\,, \qquad (3.169)$$

since (3.23) and (3.24) implies that $\frac{4\epsilon_0}{x^+}T^{(2)}_{0;(0)} = -T^{(0)}_{0;(0)}$, and (3.166) is indeed equivalent to (3.169). By the rescaling $\chi_{(+)}(x^+) = y^{-\nu'}J(y)$ brings (3.169) to Bessel's differential equation (3.112) with index $\nu' = 1-\epsilon_0$ and source $K = 4\epsilon_0 y^{\nu'-2}$ with solution (3.119) if $\nu' \neq 0, 1, \dots$. The logarithm starting at order $x^+\log x^+$ corresponds to the lowest-weight nature of $T^{(0)}_{2;(0)}$. If $\nu' = 0$ the element $T^{(0)}_{2;(0)}$ is a two-fold root to the lowest-weight condition leading to quadratic logarithms $x^+(\log x^+)^2$. Likewise, if $D = 5+2p = 7, \dots, p = 1, 2, \dots$ then the quadratic logarithms set in at order $(x^+)^{p+1}(\log x^+)^2$. We leave the analysis of these cases for future work.

3.7 Adjoint singleton-anti-singleton composites

As seen in section 2, an element $Q \in \mathfrak{ho}$ can be mapped via the twisted reflector $|\widetilde{\mathbb{1}}\rangle_{12}$ to the state $|\widetilde{Q}\rangle_{12}$ carrying the untwisted $\mathfrak{ho}$-representation $|\mathrm{ad}_Q\widetilde{Q}'\rangle_{12} = Q|\widetilde{Q}'\rangle_{12}$. Since $|\widetilde{\mathbb{1}}\rangle_{12} = k(2)|\mathbb{1}\rangle_{12}$ and $|\mathbb{1}\rangle_{12}$ contains the reduced singleton-singleton reflector $|\mathbf{M}_{\mathfrak{D}}\rangle_{12}$ given in (3.153) and (3.147), it follows that $|\widetilde{\mathbb{1}}\rangle_{12}$ contains the reduced singleton-anti-singleton reflector

$$|\widetilde{\mathbf{M}}_{\mathfrak{D}}\rangle_{12} = k(2)|\mathbf{M}_{\mathfrak{D}}\rangle_{12} \ = \ (1+\pi)\sum_{n=0}^{\infty}[\mathcal{N}_n]^{-1}|n\rangle^+_{1;r(n)}|n\rangle^-_{2;r(n)}\,, \qquad (3.170)$$

where we use the basis (3.149). One can show that $(M_{AB}(1)+M_{AB}(2))|\widetilde{\mathbf{M}}_{\mathfrak{D}}\rangle_{12} = 0$. The ℓ-th adjoint level, which is the lowest-and-highest-weight space ($s = 2\ell+2$)

$$\mathcal{L}_\ell \simeq \mathfrak{D}(-(s-1);(s-1)) \ = \bigoplus_{\substack{e\in\mathbb{Z},\, n\geqslant 0\\ j_1\geqslant j_2\geqslant 0\\ |e|\leqslant j_1-j_2\\ j_1+n\leqslant s-1}} \mathbb{C}\otimes M_{e;(j_1,j_2);n}\,, \qquad (3.171)$$

with $M_{e;(j_1,j_2);n}$ given below (3.37), is thus reflected by $|\widetilde{\mathbf{M}}_{\mathfrak{D}}\rangle_{12}$ to a bimodule containing the lowest-weight and highest-weight states

$$|\mp(s-1);(s-1)\rangle^\pm_{12;r(s-1)} = (L^\mp_{\{r_1}\cdots L^\mp_{r_{s-1}\}})(1)|\widetilde{\mathbf{M}}_{\mathfrak{D}}\rangle_{12}\,, \qquad (3.172)$$

where we note that $L^\mp_r|\mp(s-1);(s-1)\rangle_{12} = 0$ follows from the $\mathfrak{g}$-invariance of $|\widetilde{\mathbf{M}}_{\mathfrak{D}}\rangle_{12}$. Thus one has

$$|\mp(s-1);(s-1)\rangle^\pm_{12;r(s-1)} = (1+\pi)\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!(\epsilon_0)_n}|n+s-1\rangle^\mp_{1;r(s-1)t(n)}|n\rangle^\pm_{2;t(n)}\,, \qquad (3.173)$$

that is, the tensor product of a scalar singleton and a scalar anti-singleton can be expanded in terms of finite-dimensional adjoint levels, leading to the *twisted Flato-Fronsdal formula*:

$$(\mathfrak{D}^+_0\otimes\mathfrak{D}^-_0)\oplus(\mathfrak{D}^-_0\otimes\mathfrak{D}^+_0) = \bigoplus_{s=0}^{\infty}\mathfrak{D}(-(s-1);(s-1))\,. \qquad (3.174)$$

4. Conclusions

The unfolded formulation of dynamics provides a dual fiber description of standard field equations in terms of functions in the enveloping algebra of the underlying isometry group and associated finite-dimensional as well as infinite-dimensional representations. This approach is natural in the context of higher-spin gauge theories. In this paper we have used it to examine the harmonic expansion of simple bosonic higher-spin gauge theories around constantly curved backgrounds. We have focused on the case of AdS_D with the aim of assessing the basic premises (i)–(iv) listed at the end of section 1.4.

Indeed, we have confirmed properties (i)–(iii). In particular, the factorization of $\mathcal{M}^{(+)}$ has been found to be given in terms of the angleton $\mathcal{S}^{(+)}$ which is the lowest-spin module given by the left action on the static ground state $T^{(0)}_{0;(0)}$. Our nomenclature is motivated by the fact that $\mathcal{S}^{(+)}$ fills a wedge in compact weight space, as opposed to the singleton $\mathfrak{D}_0$ which fills a single line. The relation between $\mathcal{M}^{(+)}$ and the angletons (see eq. (3.32)) involves an equivalence relation $\sim$, which is a form of "gauging" needed in order to avoid degeneracy, and that has no analog in the Flato-Fronsdal formula. Moreover, although $\mathcal{M}^{(+)}$ contains $(1+\pi)(\mathfrak{D}_0 \otimes (\mathfrak{D}_0)^*)$, the angleton does not contain $(1+\pi)\mathfrak{D}_0$ nor any of the states in its negative-spin extension discussed in appendix E. The field theoretic interpretation of the angletons therefore poses an interesting problem, as does that of the half-integer one-sided spins in $\mathcal{S}^{(0)}_{e;(0)}$ for odd $e+D$ and $D>4$.

We have also found that the module $\mathcal{W}^{(+)}_{(0)}$, which contains even runaway scalar fields, is unitarizable for all D. The situation is intriguing since $\mathcal{W}^{(+)}_{(0)}$ forms an indecomposable $\mathfrak{g}$-structure together with $\mathfrak{D}(2;(0))$ which is known to be unitarizable only if $D \leqslant 7$. Moreover, the twisted-adjoint $\mathfrak{ho}$ action yields higher-spin lowest-spin modules $\mathcal{W}^{(+)}_{(s)}$ containing elements with negative rescaled trace norms such as the static ground state $T^{(1)}_{0;(1,1)}$ in $\mathcal{W}^{(+)}_{(1)}$. It might be the case, however, that the trace norm remains (negative or positive) definite for fixed s, and a decisive analysis should take into account additional phase-factors coming from the definitions of real forms of the higher-spin generators and possible internal sectors. In concluding, we also note that there exists an inequivalent inner product on $\mathcal{M}$ induced from the (non-definite) inner product on the angleton.

An important issue left out in this paper is related to the property (v), namely whether $\mathcal{M}$ can be equipped with an associative structure such that the completeness relation $\mathbb{1}_{\mathcal{M}} \star \mathbb{1}_{\mathcal{M}} = \mathbb{1}_{\mathcal{M}}$ can be imposed on the decomposition of the unity given in (3.107). This issue is related to the question whether the linearized solutions in $\mathcal{M}$ admit non-linear completions within Vasiliev's equations. Physically speaking, such solutions would describe "one-body" systems whose extensions to "two-body" systems might provide an effective potential in some limit. We plan to give more details on this in [36].

A related omitted topic is the inclusion of compact-weight elements that are singular functions on the enveloping algebra. In the case of AdS_D, each analytic compact $\mathfrak{so}(2) \oplus \mathfrak{so}(D-1)$-type $T^{(s)}_{e;(j_1,j_2)}$ is accompanied by a dual non-analytic element $\widetilde{T}^{(s)}_{e;(j_1,j_2)}$ forming a dual module $\widetilde{\mathcal{M}}$. Since $T^{(s)}_{e;(j_1,j_2)}$ corresponds to the generalized spherical harmonic function that is finite at $r=0$, its dual must correspond to the singular fluctuation field that blows up at $r = 0$. Moreover, for $D = 4$ and $s = 0$, the duals of particles have logarithmic singularities at $E = 0$, while the duals of runaway solutions have poles at $E = 0$. Thus one might speculate that the $r \leftrightarrow 1/r$ symmetry mentioned in section 1.4 corresponds to that $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ have dual indecomposable structures viewed as $\mathfrak{ho}(D+1;\mathbb{C})$-modules (as defined in appendix A), and that this $\mathbb{Z}_2$ symmetry extends to the full level. For example, on the Euclidean S^D minus the north-pole N and the south-pole S, this symmetry would exchange solitons/runaways centered at N with runaways/solitons centered at S.

Another direction to pursue is the extension of the phase-space quantization program from finite-dimensional non-compact algebras to infinite-dimensional non-compact algebras. In particular we would like to realize the spectrally flowed multipleton vertex operators of the subcritical WZW models discussed in [34] directly in terms of the Kac-Moody currents, without resorting to free-field constructions, and then study whether these constructions could be extended to the angletons.

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A. Basic properties of some infinite-dimensional $\mathfrak{so}(D+1;\mathbb{C})$ modules

This appendix contains some basic properties of various infinite-dimensional unitarizable representations of $\mathfrak{so}(D+1;\mathbb{C})$ that occur in the text. General treatises can be found for example in [45–47].

A.1 Harish-Chandra modules and Verma modules

An infinite-dimensional irreducible $\mathfrak{g}$ module $\mathfrak{R}$ can be "sliced" in many different ways. The slicing under a Cartan subalgebra $\mathfrak{g}^{(0)}$ yields the *weight-space representation* $\mathfrak{R}' = \bigoplus_\lambda \text{mult}(\lambda)\mathfrak{R}_\lambda$ where $\mathfrak{R}_\lambda = \mathbb{C} \otimes |\lambda\rangle$ with $|\lambda\rangle$ being an eigenstate of the generators in $\mathfrak{g}^{(0)}$ with eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$, $r = \dim \mathfrak{g}^{(0)}$, referred to as a weight vector, or just weight, and the multiplicities $\text{mult}(\lambda) \in \{1, 2, \ldots\} \cup \{\infty\}$. If $\mathfrak{R}$ has finite dimension then $\mathfrak{R} \simeq \mathfrak{R}'$ while this need not hold true if $\mathfrak{R}$ has infinite dimension. The slicing $\mathfrak{R}|_\mathfrak{h}$ of $\mathfrak{R}$ under a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be *admissible* if it contains only finite-dimensional $\mathfrak{h}$ irreps, sometimes referred to as $\mathfrak{h}$-types, with finite multiplicities, viz.

$$\mathfrak{R}|_\mathfrak{h} = \bigoplus_\kappa \text{mult}(\kappa)\mathfrak{R}_\kappa \,, \qquad \text{mult}(\kappa) \dim \mathfrak{R}_\kappa < \infty \,, \tag{A.1}$$

where κ are referred to as the *compact weights* of $\mathfrak{h}$ (we note that if $\mathfrak{g}^{(0)} \subset \mathfrak{h}$ then the further decomposition of $\mathfrak{R}|_\mathfrak{h}$ under $\mathfrak{g}^{(0)}$ might yield weights λ with $\text{mult}(\lambda) = \infty$). A remarkable theorem (see for example [47]) states that if $\mathfrak{R}$ is a unitary representation of a real form of

$\mathfrak{g}$ with maximal compact subalgebra $\mathfrak{h}$ then $\text{mult}(\kappa) \leqslant \dim \mathfrak{R}_\kappa$. If all $\mathfrak{h}$-types are generated by $\mathcal{U}[\mathfrak{g}]$ starting from a finite number of $\mathfrak{h}$-types then $\mathfrak{R}$ is referred to as a *Harish-Chandra module*. In particular, if $\mathfrak{R} \simeq \mathfrak{R}|_\mathfrak{h}$ we shall refer to $\mathfrak{R}$ as a $(\mathfrak{g}|\mathfrak{h})$ module.

If $\mathfrak{R} = \bigoplus_\kappa \mathfrak{R}_\kappa$ is a left $(\mathfrak{g}|\mathfrak{h})$ module with dual right module $\mathfrak{R}^* = \bigoplus_\kappa \mathfrak{R}^{*\kappa}$, then the resulting matrix representation of $\mathfrak{g}$ in $\mathfrak{R}$ and $\mathfrak{R}^*$ can be written as $\rho(X) = \sum_{\kappa,\kappa'} v_\kappa \otimes v^{*\kappa'} \rho^\kappa{}_{\kappa'}(X)$. Equipping $\mathfrak{R}$ with the symmetric bilinear inner product $\eta_{\kappa\kappa'} \equiv (v_\kappa, v_{\kappa'})|_\mathfrak{R} \equiv \delta_{\kappa\kappa'}(v_\kappa, v_{\kappa'})_{\mathfrak{R}_\kappa}$, so that $\mathfrak{R}$ and $\mathfrak{R}^*$ become isomorphic as vector spaces, makes $\mathfrak{R}^*$ equivalent to the *dual* left module $\widetilde{\mathfrak{R}} = \bigoplus_\kappa \widetilde{\mathfrak{R}}_\kappa$ with representation matrix $\widetilde{\rho}(X) = -(\eta\rho(X)\eta^{-1})^T$.

A reducible $\mathfrak{g}$ module $\mathfrak{R}$ is said to be *indecomposable* if it contains a proper ideal $\mathfrak{I}$ whose complement is not invariant. This is denoted by $\mathfrak{R} = \mathfrak{R}_1 \inplus \mathfrak{R}_2$ where $\mathfrak{R}_1 \simeq \mathfrak{I}$ and $\mathfrak{R}_2 \simeq \mathfrak{R}/\mathfrak{I}$. Thus, the representation of $\mathfrak{g}$ in $\mathfrak{R}$ is of the form $\rho = \begin{bmatrix} \rho_1 & \alpha_{12} \\ 0 & \rho_2 \end{bmatrix}$, where ρ_i are the representations of $\mathfrak{g}$ in $\mathfrak{R}_i$, $i = 1, 2$, and the off-diagonal piece $\alpha_{12} : \mathfrak{R}_2 \to \mathfrak{R}_1$, referred to as a cocycle, obeys $\rho_1(X)\alpha_{12}(Y) + \alpha_{12}(X)\rho_2(Y) - (X \leftrightarrow Y) = \alpha_{12}([X,Y])$ for all $X, Y \in \mathfrak{g}$. It follows that $t\alpha_{12}$ with $t \in \mathbb{C}$ is a cocycle of a rescaled indecomposable structure ρ^t, such that ρ^0 is decomposable, and that the dual left module $\widetilde{\mathfrak{R}} = \widetilde{\mathfrak{R}}_1 \niplus \widetilde{\mathfrak{R}}_2$ carries the dual indecomposable structure $\widetilde{\rho} = \begin{bmatrix} \widetilde{\rho}_1 & 0 \\ \widetilde{\alpha}_{21} & \widetilde{\rho}_2 \end{bmatrix}$ with $\widetilde{\alpha}_{21}(X) = -(\eta_1\alpha_{12}(X)\eta_2^{-1})^T$.

A *highest-weight* module $\mathfrak{D}(\Lambda)$ is a $(\mathfrak{g}|\mathfrak{g}^{(0)})$ module in which the weights are bounded from above by a highest weight Λ in the sense that the corresponding (unique) highest-weight state $|\Lambda\rangle \in \mathfrak{D}(\Lambda)$ can be reached by successive maximization of λ_k keeping $\lambda_{k'}$, $k' < k$ maximal, inducing a weak three-grading $\mathfrak{g} = \mathfrak{g}^{(-)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+)}$ wherein $\mathfrak{g}^{(0)}$ is the Cartan subalgebra, $\mathfrak{g}^{(+)}|\Lambda\rangle = 0$, $[\mathfrak{g}^{(\pm)}, \mathfrak{g}^{(\pm)}] \subseteq \mathfrak{g}^{(\pm)}$ and $[\mathfrak{g}^{(0)}, \mathfrak{g}^{(\pm)}] \subseteq \mathfrak{g}^{(\pm)}$. The corresponding infinite-dimensional *Verma module*, or cyclic $(\mathfrak{g}|\mathfrak{g}^{(0)})$ module, $\mathfrak{V}(\Lambda) \equiv \mathcal{U}[\mathfrak{g}^{(-)}]|\Lambda\rangle$ (where we recall that $\mathcal{U}[\mathfrak{g}]$ denotes the enveloping algebra of a Lie algebra $\mathfrak{g}$) is irreducible for generic Λ and indecomposable for critical Λ in which case it contains at least one excited state that is annihilated by $\mathfrak{g}^{(+)}$, referred to as a *singular vector*. The $\mathcal{U}[\mathfrak{g}^{(-)}]$ action on the singular vectors generates a maximal ideal $\mathfrak{N}(\Lambda) \subset \mathfrak{V}(\Lambda)$, sometimes referred to as the null module, and

$$\mathfrak{V}(\Lambda) = \mathfrak{D}(\Lambda) \niplus \mathfrak{N}(\Lambda)\,, \qquad \mathfrak{D}(\Lambda) = \frac{\mathfrak{V}(\Lambda)}{\mathfrak{N}(\Lambda)}\,. \tag{A.2}$$

In the case of $\mathfrak{g} = \mathfrak{so}(D+1;\mathbb{C})$ the finite-dimensional representations are *highest-and-lowest-weight* spaces. In Euclidean signature $\eta_{AB} = (+, \dots, +)$ and with $(M_{D+3-2k,D+2-2k} - \lambda_k) \mid \lambda\rangle = 0$ for $k = 1, \dots, \nu = [(D+1)/2]$, these arise for highest weights obeying

$$\Lambda \in \mathbb{Z}^r \cup \left(\mathbb{Z} + \frac{1}{2}\right)^r, \qquad \Lambda_1 \geqslant \Lambda_2 \geqslant \cdots \geqslant \Lambda_\nu \geqslant 0\,, \tag{A.3}$$

which we shall refer to as positive integer or positive half-integer highest weights, or more shortly, integer or half-integer $\mathfrak{g}$-spins. They correspond to traceless tensors or gamma-traceless tensor-spinors, respectively, in shapes with Λ_k cells in the kth row (as can be seen by going to helicity basis). In the tensorial case the lowest-weight state is $|-\Lambda\rangle$, thus obeying $\mathfrak{g}^{(-)} \mid -\Lambda\rangle = 0$.

A.2 Lowest-weight representations

Upon Wick-rotating the D and $D+1$ directions into time-like 0 and $0'$ directions and going to the helicity basis $V^{\pm} = V^0 \mp iV^{0'}$ and $V_{\pm} = \frac{1}{2}(V_0 \mp iV_{0'})$ with $\eta^{+-} = -2$, one identifies the energy operator and the energy-raising and energy-lowering ladder operators

$$E = -M_{D+1,D} \;=\; M_{0'0} \;=\; iM_+{}^+ \;=\; -2iM_{+-}\,, \tag{A.4}$$

$$L_r^{\pm} = M_{0r} \mp iM_{0'r} \;=\; M_{0r} \mp iP_r \;=\; M_r{}^{\pm} \;=\; -2M_{r\mp}\,, \tag{A.5}$$

with $P_a = M_{0'a} = (E, P_r)$. The resulting commutation rules read

$$[L_r^-, L_s^+] = 2iM_{rs} + 2\delta_{rs}E\,, \qquad [E, L_r^{\pm}] \;=\; \pm L_r^{\pm}\,, \qquad [M_{rs}, L_t^{\pm}] \;=\; 2i\delta_{t[s}L_{r]}^{\pm}\,, \tag{A.6}$$

and $[M_{rs}, M_{tu}] = 4i\delta_{[t|[s}M_{r]|u]}$, exhibiting a three-grading whereby $\mathfrak{g} = \mathfrak{l}^- \oplus (\mathfrak{e} \oplus \mathfrak{s}) \oplus \mathfrak{l}^+$ such that $[E, \mathfrak{l}^{\pm}] = \pm\mathfrak{l}^{\pm}$ and hence $[\mathfrak{l}^{\pm}, \mathfrak{l}^{\pm}] = 0$. In this basis, the $\mathfrak{g}$-automorphism π defined in (2.22) acts as $\pi(L_r^{\pm}) = L_r^{\mp}$, $\pi(E) = -E$ and $\pi(M_{rs}) = M_{rs}$, and extends to a map between $(\mathfrak{g}|\mathfrak{e}\oplus\mathfrak{s})$ modules as the reflection $\pi(|e; \overrightarrow{s}\rangle) = |-e; \overrightarrow{s}\rangle$. The $\mathfrak{g}$-antiautomorphism τ defined in (2.2) extends to the reflection

$$\tau(|e; \overrightarrow{s}\rangle) = \varphi(e; \overrightarrow{s})\langle -e; \overrightarrow{s}|\,, \qquad \tau^2 \;=\; \begin{cases} \mathrm{Id}, & \text{integer spin} \\ -\mathrm{Id}, & \text{half-integer spin} \end{cases}\,, \tag{A.7}$$

where $\langle -e; \overrightarrow{s}|$ belong to dual $(\mathfrak{g}|\mathfrak{e}\oplus\mathfrak{s})$ modules and $\varphi(e; \overrightarrow{s})$ are phase factors.

One particular type of infinite-dimensional representations of $\mathfrak{so}(D+1;\mathbb{C})$ are highest-weight spaces in which $\overrightarrow{s}_0 \equiv (\Lambda_2, \dots, \Lambda_\nu)$ is a spin of $\mathfrak{s} \simeq \mathfrak{so}(D-1;\mathbb{C})$, while the lowest eigenvalue $e_0 \equiv -\Lambda_1$ of the energy operator $E = -M_{D+1,D}$ is no longer quantized. Splitting $\mathfrak{g}^{(\pm)} = \mathfrak{s}^{(\pm)} \oplus \mathfrak{l}^{\mp}$, where $\mathfrak{s}^{(\pm)} \in \mathfrak{s} \cap \mathfrak{g}^{(\pm)}$ and $\mathfrak{l}^{\pm}$ are the "effective" ladder operators obeying $[\mathfrak{l}^{\pm}, \mathfrak{l}^{\pm}] = 0$, it follows that $\mathfrak{V}(\Lambda) = \mathcal{U}[\mathfrak{l}^+]\mathcal{U}[\mathfrak{s}^{(-)}]|\Lambda\rangle$, where the $\mathfrak{s}$-submodule $\mathcal{U}[\mathfrak{s}^{(-)}]|\Lambda\rangle$ contains a null $\mathfrak{s}$-module $\mathfrak{N}(\mathfrak{s}|\overrightarrow{s}_0)$ such that $\mathfrak{C}_{e_0;\overrightarrow{s}_0} \equiv \frac{\mathcal{U}[\mathfrak{s}^{(-)}]|e_0;\overrightarrow{s}_0\rangle}{\mathfrak{N}(\mathfrak{s}|\overrightarrow{s}_0)}$ is the $(\mathfrak{e}\oplus\mathfrak{s})$-type with (minimal) energy e_0. Factoring out the generic $\mathfrak{g}$ ideal $\mathcal{U}[\mathfrak{l}^+]\mathfrak{N}(\mathfrak{s}|\overrightarrow{s}_0) \subseteq \mathfrak{N}(\Lambda)$ from $\mathfrak{V}(\Lambda)$ yields the *generalized Verma module*

$$\mathfrak{C}(e_0; \overrightarrow{s}_0) = \frac{\mathfrak{V}(\Lambda)}{\mathcal{U}[\mathfrak{l}^+]\mathfrak{N}(\mathfrak{s}|\overrightarrow{s}_0)} \;=\; \mathcal{U}[\mathfrak{l}^+]\,\mathfrak{C}_{e_0;\overrightarrow{s}_0} \;=\; \bigoplus_{e\geqslant e_0;\overrightarrow{s}} \mathrm{mult}(e;\overrightarrow{s})\,\mathfrak{C}_{e;\overrightarrow{s}}\,, \tag{A.8}$$

which is thus a $(\mathfrak{g}|\mathfrak{e}\oplus\mathfrak{s})$ module with energies bounded from below by e_0. Thus we have a *lowest-energy* state $|e_0; \overrightarrow{s}_0\rangle^+ \equiv |(\Lambda_1, \overrightarrow{s}_0)\rangle$ and a *highest-energy* state $|-e_0; \overrightarrow{s}_0\rangle^- \equiv \pi(|e_0; \overrightarrow{s}_0\rangle$ obeying

$$(E \mp e_0)|\pm e_0; \overrightarrow{s}_0\rangle^{\pm} \;=\; 0\,, \qquad L_r^{\mp}|\pm e_0; \overrightarrow{s}_0\rangle^{\pm} \;=\; 0\,, \tag{A.9}$$

and the restriction of $\mathfrak{V}(\Lambda)$ and $\pi(\mathfrak{V}(\Lambda))$, respectively, to the subspace $\mathcal{U}[\mathfrak{l}^{\pm}]|\pm e_0; \overrightarrow{s}_0\rangle^{\pm}$ yields

$$\mathfrak{C}^{\pm}(\pm e_0; \overrightarrow{s}_0) = \bigoplus_{\pm e\geqslant e_0;\overrightarrow{s}} \mathbb{C}\otimes|e; \overrightarrow{s}\rangle^{\pm}\,, \qquad |e_0+n; \overrightarrow{s}\rangle = \mathbf{P}_{\overrightarrow{s}}\left[(L^{\pm})^n|\pm e_0; \overrightarrow{s}_0\rangle^{\pm}\right]\,, \tag{A.10}$$

where, in a slight abuse of notation, we are denoting $\mathfrak{C}_{e;\vec{s}}$ by $|e;\vec{s}\rangle$. The singular vectors are states $|e'_0;\vec{s}'_0\rangle^\pm$ with $\pm e'_0 > e_0$ obeying $L^\mp_r|e'_0;\vec{s}'_0\rangle^\pm = 0$. They generate generalized Verma submodules forming a maximal ideal $\mathfrak{I}^\pm(\pm e_0;\vec{s}_0)$ and

$$\mathfrak{D}^\pm(\pm e_0;\vec{s}_0) = \frac{\mathfrak{C}^\pm(\pm e_0;\vec{s}_0)}{\mathfrak{I}^\pm(\pm e_0;\vec{s}_0)}\,, \tag{A.11}$$

where thus $\mathfrak{D}^+(e_0;\vec{s}_0) = \mathfrak{D}(\Lambda_1,\vec{s}_0)$ and $\mathfrak{D}^-(-e_0;\vec{s}_0) \equiv \pi(\mathfrak{D}(\Lambda))$. For generic e_0 the energy spectra are unbounded in $\mathfrak{D}^+(e_0;\vec{s}_0)$ and $\mathfrak{D}^-(-e_0;\vec{s}_0)$, that are then distinct albeit isomorphic representations which we shall refer to as lowest-weight and highest-weight spaces, respectively (although they are actually of opposite type back in Euclidean signature). In this context the finite-dimensional representations are referred to as lowest-and-highest-weight spaces.[31]

The lowest-weight and highest-weight representations are unitarizable in two-time signature for $e_0 \geqslant e_0(\vec{s}_0)$ in such a way that $\mathfrak{D}^\pm(e_0;\vec{s}_0) \simeq \mathfrak{C}^\pm(-e_0;\vec{s}_0)$ for $e_0 > e_0(\vec{s}_0)$, where $e_0(\vec{s}_0)$ is a critical value at which at least one singular vector develops while E remains unbounded from above (two singular vectors arise in conformal cases with $s_0 \geqslant 1$). The critical cases of interest to us are:

$$\textit{scalar and spinor singletons}: \qquad e_0 = s_0 + \epsilon_0\,, \qquad s_0 = 0, \frac{1}{2}\,, \tag{A.12}$$

$$\textit{composite massless particles}: \qquad e_0 = s_0 + 2\epsilon_0\,, \qquad s_0 = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\,, \tag{A.13}$$

where $\epsilon_0 = \frac{1}{2}(D-3)$. The corresponding singular vectors are:

$$\textit{singletons} \quad s_0 = 0: \qquad |\epsilon_0+2;(0)\rangle = L^+_r L^+_r|\epsilon_0;(0)\rangle\,, \tag{A.14}$$

$$s_0 = \frac{1}{2}: \qquad |\epsilon_0+\frac{3}{2};\left(\frac{1}{2}\right)\rangle_i = (\gamma_r)_i{}^j L^+_r|\epsilon_0+\frac{1}{2};\left(\frac{1}{2}\right)\rangle_j\,, \tag{A.15}$$

$$\textit{massless} \quad s_0=1,2,\ldots: \quad |s_0+2e_0+1;(s_0-1)\rangle_{r(s_0-1)} = L^+_t|s_0+2e_0;(s_0)\rangle_{tr(s_0-1)}\,, \tag{A.16}$$

$$s_0=\frac{3}{2},\frac{5}{2},\cdots: \qquad |s_0+2e_0+1;(s_0-1)\rangle_{i,r(s_0-\frac{3}{2})} = L^+_t|e_0;(s_0)\rangle_{i,tr(s_0-\frac{3}{2})}\,, \tag{A.17}$$

where the $\mathfrak{s}$ irreps of the ground states are given by

$$M_{rs}|e_0;(s_0)\rangle_{t(s_0)} = 2is_0\delta_{t1[s}|e_0;(s_0)\rangle_{r]t(s_0-1)}\,, \tag{A.18}$$

$$M_{rs}|e_0;(s_0)\rangle_{i,t(s_0-\frac{1}{2})} = 2i\left(s_0-\frac{1}{2}\right)\delta_{t1[s}|e_0;(s_0)\rangle_{i,r]t(s_0-\frac{3}{2})} - \frac{i}{2}(\gamma_{rs})_i{}^j|e_0;(s_0)\rangle_{j,t(s_0-\frac{1}{2})}\,, \tag{A.19}$$

with γ_r and $\gamma_{rs} = \gamma_{[r}\gamma_{s]}$ being $\mathfrak{so}(D-1)$ Dirac matrices, and the right-hand sides are automatically traceless and γ-traceless, respectively. The phase factors in (A.7) can be chosen to be

$$\textit{scalar singletons}: \qquad \tau(|\pm\epsilon_0;(0)\rangle^\pm) = {}^\mp\langle\mp\epsilon_0;(0)|\,, \tag{A.20}$$

[31] The finite-dimensional irreps arise for negative e_0. For example, the $(D+1)$-plet $\mathfrak{D}^-(-1;(0)) = \mathfrak{C}^-(-1;(0))/\mathfrak{I}(-1;(0))$. The singular vector $L^+_{\{r}L^+_{s\}}|-1;(0)\rangle$ generates $\mathfrak{I}(-1;(0)) \simeq \mathfrak{C}(1;(2))$ containing *all* states in $\mathfrak{C}^-(-1;(0))$ with energy $e \geqslant 2$ since $L^-_r L^+_s L^+_t L^+_t|-1;(0)\rangle = -6L^+_{\{r}L^+_{s\}}|-1;(0)\rangle$. Thus $\mathfrak{D}(-1;(0))$ is spanned by $|-1;(0)\rangle$, $|0;(1)\rangle = L^+_r|-1;(0)\rangle$ and $|1;(0)\rangle = L^+_r L^+_r|-1;(0)\rangle$, and the highest-weight state is $L^+_r|1;(0)\rangle \in \mathfrak{I}(-1;(0))$.

$$\textit{massless bosons}: \qquad \tau(|\pm(s_0+2\epsilon_0);(s_0)\rangle^{\pm}) \;=\; (-)^{s_0}\ {}^{\mp}\langle\mp(s_0+2\epsilon_0);(s_0)|\,, \tag{A.21}$$

$$4D\ \textit{spinor singletons}: \qquad \tau\left(|\pm 1;\left(\frac{1}{2}\right)\rangle^{\pm}\right) \;=\; i\ {}^{\mp}\langle\mp 1;\left(\frac{1}{2}\right)|\,, \tag{A.22}$$

$$4D\ \textit{massless fermions}: \qquad \tau(|\pm(s_0+1);(s_0)\rangle^{\pm}) = (-)^{s_0-\frac{1}{2}}i\ {}^{\mp}\langle\mp(s_0+1);(s_0)|\ . \tag{A.23}$$

A.3 Lowest-spin modules

In this paper, we refer to an infinite-dimensional $(\mathfrak{g}|\mathfrak{e}\oplus\mathfrak{s})$ module $\mathcal{W}(e_0;\overrightarrow{s_0})$ as a *lowest-spin module* if:

(i) the energy spectrum is *unbounded* from both above and below (while the spins are bounded from below at least by $(0\dots0)$ or $(\frac{1}{2}\dots\frac{1}{2})$ by the admissibility assumption); and

(ii) $\mathcal{W}(e_0;\overrightarrow{s_0}) = \mathcal{M}(e_0;\overrightarrow{s}_0)/\mathcal{I}(e_0;\overrightarrow{s}_0)$ where $\mathcal{M}(e_0;\overrightarrow{s}_0) = \mathcal{U}[\mathfrak{l}^-]\mathcal{U}[\mathfrak{l}^+]|e_0;\overrightarrow{s}_0\rangle$ with the *static ground state* $|e_0;\overrightarrow{s}_0\rangle$ being the $(\mathfrak{e}\oplus\mathfrak{s})$-type with *minimal* $|e|+\dim(\overrightarrow{s}_0)$, and $\mathcal{I}(e_0;\overrightarrow{s}_0)$ is the maximal ideal.

The state generation yields a canonical inner product $(\cdot,\cdot)_{\mathcal{W}(e_0;\overrightarrow{s}_0)}$ (following the prescription under (3.84)). We say that $\mathcal{W}(e_0;\overrightarrow{s}_0)$ is *unitary* if $(\cdot,\cdot)_{\mathcal{W}(e_0;\overrightarrow{s}_0)}$ is positive definite for the real form of $\mathcal{W}(e_0;\overrightarrow{s}_0)$ compatible with the $\mathfrak{g}$ action. We note that $L^{\pm}_r|e_0;\overrightarrow{s}_0\rangle$ are non-vanishing, and that $\mathcal{M}(e_0;\overrightarrow{s}_0)$ may contain an infinite number of $(\mathfrak{e}\oplus\mathfrak{s})$-types with given energy e, in which case its decomposition under the $\mathfrak{g}^{(0)}\subset\mathfrak{e}\oplus\mathfrak{s}$ contains infinitely degenerate weights.

A.4 Twisted-adjoint $(\mathfrak{g}|\mathfrak{m}_1)$ modules

The twisted-adjoint $(\mathfrak{g}|\mathfrak{m}_1)$ modules (where we follow the notation of sections 2.1 and 2.2)

$$\mathfrak{R}(s) = \bigoplus_{k\geqslant 0}\mathfrak{R}_{(s+k,s)}\,, \qquad \operatorname{mult}(s+k,s) \;=\; 1\,, \qquad s=0,1,\dots\,, \tag{A.24}$$

with representation matrices given by

$$M_{mn}|^{(s,s)}_{(s_1,s_2)}\rangle_{p(s_1),q(s_2)} = 2is_1\eta_{p_1[n}|^{(s,s)}_{(s_1,s_2)}\rangle_{m]p(s_1-1),q(s_2)} + 2is_2\eta_{q_1[n|}|^{(s,s)}_{(s_1,s_2)}\rangle_{p(s_1),|m]q(s_2-1)}\,, \tag{A.25}$$

$$P_m|^{(s,s)}_{(s+k,s)}\rangle_{n(s+k),p(s)} = 2\Delta_{s+k,s}|^{(s,s)}_{(s+k+1,s)}\rangle_{m\{n(s+k),p(s)\}} + 2\lambda^{(s)}_k\eta_{m\{n_1}|^{(s,s)}_{(s+k-1,s)}\rangle_{n(s+k-1),p(s)\}}\,, \tag{A.26}$$

where the traceless type-$(s+k,s)$ projection implies that $\Delta_{s+k,s} = \frac{(k+2)(k+s+1)}{(k+1)(k+s+2)}$ and[32]

$$\eta_{m\{n_1}|^{(s,s)}_{(s+k-1,s)}\rangle_{n(s+k-1),p(s)\}} = \eta_{mn_1}|^{(s,s)}_{(s+k-1,s)}\rangle_{n(s+k-1),p(s)}$$

[32] To compute $\Delta_{s+k,s}$ one may use $\mathbf{P}(P_cT_{a(s+k),b(s)}) = \Delta_{s+k,s}\mathbf{P}T_{c\{a(s+k),b(s)\}} = \Delta_{s+k,s}T_{ca(s+k),b(s)}$ with $\mathbf{P}\equiv\mathbf{P}_{\langle ca(s+k),b(s)\rangle}$, or equivalently $T_{a(s+k+1),b(s)} = \Delta_{s+k,s}T_{a\langle a(s+k),b(s)\rangle}$ where

$$T_{a\langle a(s+k),b(s)\rangle} \;=\; \frac{(s+k)!s!}{(s+k+1)\cdots(k+2)(k)\cdots1\times s!}\sum_{n=0}^{s}(-1)^n\binom{s}{n}T_{a(s+k+1-n)b(n),a(n)b(s-n)}\ .$$

JHEP10(2008)022

$$
\begin{aligned}
&+\alpha_{s+k,s}\eta_{n(2)}|{}^{(s,s)}_{(s+k-1,s)}\rangle_{n(s+k-2)m,p(s)}+\\
&+\beta_{s+k,s}\eta_{n(2)}|{}^{(s,s)}_{(s+k-1,s)}\rangle_{n(s+k-2)p,mp(s-1)}\\
&+\gamma_{s+k,s}\eta_{np}T_{n(s+k-1),mp(s-1)}\,, \qquad\qquad \text{(A.27)}
\end{aligned}
$$

with $\alpha_{s+k,s} = -\frac{1}{2}\frac{s+k-1}{s+k+\epsilon_0-\frac{1}{2}}$, $\beta_{s+k,s} = \frac{1}{2}\frac{(s+k-1)s}{(s+k+\epsilon_0-\frac{1}{2})(2s+k+2\epsilon_0-1)}$ and $\gamma_{s+k,s} = -\frac{s}{2s+k+2\epsilon_0-1}$, while $\lambda_k^{(s)}$ is fixed by the closure relation $[P_m,P_n]=i\epsilon M_{mn}$, $\epsilon=\pm1$. One solution is given by the dimensional reduction of (2.12) which takes the form[33]

$$
\lambda_k^{(s)} = \frac{\epsilon}{8}\frac{k(k+s+1)(k+2s+2\epsilon_0-1)}{k+s+\epsilon_0+\frac{1}{2}}\,,\quad C_2[\mathfrak{g}|(s)] = C_2[\ell]\,,\quad s=2\ell+2\,. \qquad \text{(A.28)}
$$

For $s\geqslant 1$ this is the unique solution while $s=0$ admits the *massive* deformation[34]

$$
\lambda_k^{(0)} = \frac{\epsilon}{8}\frac{k(k^2+2\epsilon_0 k-2\epsilon_0-1-\epsilon M_0^2)}{k+\epsilon_0+\frac{1}{2}}\,,\quad C_2[\mathfrak{g}|(0)] = \epsilon M_0^2\,. \qquad \text{(A.29)}
$$

This can be seen by expanding $[P_m,P_n]|(k)\rangle_{p(k)}$ as

$$
\begin{aligned}
&2P_m|(k+1)\rangle_{np(k)} + 2\lambda_k^{(0)}\eta_{n\{p_1|}P_m|(k-1)\rangle_{|p(k-1)\}} - (m\leftrightarrow n) \qquad \text{(A.30)}\\
&= 4\lambda_{k+1}^{(0)}\eta_{m\{n}|(k)\rangle_{p(k)\}} + 4\lambda_k^{(0)}\eta_{n\{p_1}|(k)\rangle_{p(k-1)\}m}\\
&\quad +4\lambda_k^{(0)}\lambda_{k-1}^{(0)}\eta_{n\{p_1|}\eta_{m|p_2}|(k-2)\rangle_{|p(k-2)\}} - (m\leftrightarrow n)\,,
\end{aligned}
$$

where the anti-symmetrization on m and n removes the last term,[35] leaving

$$
\begin{aligned}
&4\lambda_{k+1}^{(0)}\left(\eta_{m(n}|(k)\rangle_{p(k))} + \alpha_{k+1}\eta_{(np_1}|(k)\rangle_{p(k-1))m}\right)+\\
&+4\lambda_k^{(0)}\left(\eta_{np_1}|(k)\rangle_{p(k-1)m} + \alpha_k\eta_{p(2)}|(k)\rangle_{p(k-2)nm}\right) - (m\leftrightarrow n)
\end{aligned}
$$

Using $T_{(a(s+k+1-n)b(n),a_1)a(n-1)b(s-n)} = 0$ to cycle the b-indices yields $T_{a(s+k+1-n)b(n),a(n)b(s-n)} = -\frac{n}{s+k-n+2}T_{a(s+k-n+2)b(n-1),a(n-1)b(s-n+1)} = \frac{(-1)^n}{\binom{s+k+1}{n}}T_{a(s+k+1),b(s)}$. Thus $[\Delta_{s+k,s}]^{-1} = \frac{k+1}{s+k+1}\sum_{n=0}^{s}\frac{\binom{s}{n}}{\binom{s+k+1}{n}} = \frac{(k+1)(s+k+2)}{(k+2)(s+k+1)}$ as a consequence of the lemma $\sum_{n=0}^{s}\frac{\binom{s}{n}}{\binom{s+k+1}{n}} = \frac{1}{(s+1)_{k+1}}\sum_{n=0}^{s}(s+1-n)_{k+1}$.

[33] Acting on a Lorentz tensor the mass operator $\nabla^2 = M^2 \equiv -P^2 = \epsilon(C_2[\mathfrak{m}_1]-C_2[\mathfrak{g}])$. The value of $C_2[\mathfrak{g}]$ in (A.28) yields the critical masses for composite massless Weyl tensors and scalars as examined in appendix D.

[34] The finite-dimensional highest-weight representation $\mathfrak{D}(\ell)$, containing the scalar spherical harmonics, i.e. the Killing-normalizable solutions to $(\nabla^2_{SD}+C_2[\mathfrak{so}(D+1)]|(\ell)])\phi=0$, arises inside $\mathfrak{R}(0)$ for $C_2[\mathfrak{so}(D+1)|(\ell)] = \ell(\ell+2(\epsilon_0+1))$ and $\epsilon=-1$, leading to $\lambda_k^{(0)} = \frac{1}{8}\frac{k}{k+\epsilon_0+\frac{1}{2}}(\ell+1-k)(\ell+k+2\epsilon_0+1)$, which are positive for $k=0,\dots,\ell$, vanish for $k=\ell+1$, and are negative for $k=\ell+2,\ell+3,\dots$, so that $\mathfrak{I}(0)\equiv\bigoplus_{k=\ell+1}^{\infty}\mathfrak{R}_{(k)}$ is an (non-unitarizable) ideal and $\mathfrak{D}(\ell)\equiv\mathfrak{R}(0)/\mathfrak{I}(0)=\bigoplus_{k=0}^{\ell}\mathfrak{R}_{(k)}$ a unitarizable quotient.

[35] Explicitly, using (B.12) one finds that $\eta_{n\{p_1|}\eta_{m|p_2}|(k-2)\rangle_{|p(n-2)\}} - (m\leftrightarrow n)$ equals

$$
\alpha_{k-1}\eta_{np_1}\eta_{p(2)}|(k-2)\rangle_{p(k-3)m} + \alpha_k\eta_{p(2)}\left(\eta_{m(p_3}|(k-2)\rangle_{p(k-3)n)} + \alpha_{k-1}\eta_{(p(2)}|(k-2)\rangle_{p(k-4)n)m}\right) - (m\leftrightarrow n)
$$

$$
= 2\left(\alpha_{k-1} - \frac{k-2}{k-1}\alpha_k + \frac{2\alpha_k\alpha_{k-1}}{k-1}\right)\eta_{p(2)}\eta_{p_3[n}|(k-2)\rangle_{m]p(k-3)} = 0\,.
$$

$$= 8\left(\frac{k\lambda^{(0)}_{k+1}}{k+1} - \frac{2\alpha_{k+1}\lambda^{(0)}_{k+1}}{k+1} - \lambda^{(0)}_k\right)\eta_{p_1[m}|(k)\rangle_{n]p(k-1)} \;\equiv\; 2k\eta_{p_1[m}|(k)\rangle_{n]p(k-1)}\,, \quad \text{(A.31)}$$

which is an inhomogeneous first-order recursion relation for $\lambda^{(0)}_k$ with initial datum $\lambda^{(0)}_0 = 0$ whose general solution is given by (A.29). The massively deformed twisted-adjoint spinor $\mathfrak{R}(\frac{1}{2}) = \bigoplus_{k=0}^{\infty}\mathfrak{R}_{(\frac{1}{2}+k,\frac{1}{2},\dots,\frac{1}{2})}$ has the representation matrices

$$M_{mn}|\left(k+\frac{1}{2}\right)\rangle_{p(k)} = 2ik\eta_{p_1[n}|\left(k+\frac{1}{2}\right)\rangle_{m]p(k-1)} - \frac{i}{2}\gamma_{mn}|\left(k+\frac{1}{2}\right)\rangle_{p(k)}\,, \quad \text{(A.32)}$$

$$P_m|\left(k+\frac{1}{2}\right)\rangle_{n(k)} = 2\Bigg(|\left(k+\frac{3}{2}\right)\rangle_{m\{n(k)\}} + \lambda^{(\frac{1}{2})}_k\eta_{m\{n_1}|\left(k-\frac{1}{2}\right)\rangle_{n(k-1)\}} + i\mu^{(\frac{1}{2})}_k\gamma_m|\left(k+\frac{1}{2}\right)\rangle_{\{n(k)\}}\Bigg)\,, \quad \text{(A.33)}$$

where the Dirac matrices obey $\{\gamma_m,\gamma_n\} = 2\eta_{mn}$; the γ-traceless projections

$$\eta_{m\{n_1}|\left(k-\frac{1}{2}\right)\rangle_{n(k-1)\}} = \eta_{mn_1}|\left(k-\frac{1}{2}\right)\rangle_{n(k-1)} + A_k\eta_{n(2)}|\left(k-\frac{1}{2}\right)\rangle_{n(k-2)m} + B_k\gamma_{mn_1}|\left(k-\frac{1}{2}\right)\rangle_{n(k-1)}\,, \quad \text{(A.34)}$$

$$\gamma_m|\left(k+\frac{1}{2}\right)\rangle_{\{n(k)\}} = \gamma_m|\left(k+\frac{1}{2}\right)\rangle_{n(k)} + C_k\gamma_{n_1}|\left(k+\frac{1}{2}\right)\rangle_{n(k-1)m}\,, \quad \text{(A.35)}$$

with $A_k = -\frac{k-1}{2(k+\epsilon_0)}$, $B_k = \frac{1}{2(k+\epsilon_0)}$ and $C_k = -\frac{k}{k+\epsilon_0+\frac{1}{2}}$; and the closure is solved by

$$\lambda^{(\frac{1}{2})}_k = \frac{\epsilon}{8}\frac{k(k+\epsilon_0)((k+\epsilon_0+\frac{1}{2})^2 - \epsilon M^2_{1/2})}{(k+\epsilon_0+\frac{1}{2})^2}\,,$$

$$\mu^{(\frac{1}{2})}_k = \frac{\sqrt{\epsilon}}{4}\frac{M_{1/2}}{k+\epsilon_0+\frac{3}{2}}\,, \qquad C_2\left[\mathfrak{g}|\left(\frac{1}{2}\right)\right] = \epsilon M^2_{1/2} - \frac{1}{2}(\epsilon_0+1)\left(\epsilon_0+\frac{3}{2}\right)\,. \quad \text{(A.36)}$$

A.5 The conformal twisted-adjoint modules

The conformal twisted-adjoint $(\mathfrak{g}|\mathfrak{m}_1)$ modules, namely $\mathfrak{R}(s,s) \simeq \mathfrak{D}(s+1;(s,s))$ for $s \geqslant 1$ in $D = 4$, and $\mathfrak{R}(0) \simeq \mathfrak{D}(\epsilon_0+\frac{1}{2};(0))$ and $\mathfrak{R}(\frac{1}{2}) \simeq \mathfrak{D}(\epsilon_0+1;(\frac{1}{2}))$ in $D \geqslant 3$ with the conformal masses given in (D.11), are singleton lowest-weight spaces[36] of the $\mathfrak{so}(D+2;\mathbb{C})$ generated by $M_{\underline{AB}} = (M_{AB}, R_B)$, $\eta_{\underline{AB}} = (\eta_{AB},+)$, where the conformal translations $R_A = (R_m, R)$, which obey $[M_{AB}, R_C] = 2i\eta_{C[B}R_{A]}$ and $[R_A, R_B] = -iR_{AB}$, can be represented for integer s by

$$\frac{1}{2}R_m|^{(s,s)}_{(s+k,s)}\rangle_{n(s+k),p(s)} = \Delta_{s+k,s}|^{(s,s)}_{(s+k+1,s)}\rangle_{m\{n(s+k),p(s)\}} - \lambda^{(s)}_k\eta_{m\{n_1}|^{(s,s)}_{(s+k-1,s)}\rangle_{n(s+k-1),p(s)\}}\,, \quad \text{(A.37)}$$

[36] Conversely, the $\mathfrak{g}$ singletons remain irreducible under $\mathfrak{m} \subset \mathfrak{g}$. Thus the $\mathfrak{g}$ singletons are zero-momentum representations of $\mathfrak{iso}(D-1,1)$, i.e. infinite-dimensional unitary representations of $\mathfrak{so}(D-1,1)$ with vanishing momentum. In particular, the massless UIRs of $\mathfrak{iso}(3,1)$ are conformal and as such composite in terms of *Di* $\oplus$ *Rac*, although in a non-tensorial split (see comment in appendix G.2). This suggests a manifestly $\mathfrak{so}(3,1)$-covariant unfolded realization of the singletons in the singular light-like geometry of the boundary of 4D Minkowski spacetime.

$$R|^{(s,s)}_{(s+k,s)}\rangle_{n(s+k),p(s)}=i\Delta^{(s)}_k|^{(s,s)}_{(s+k,s)}\rangle_{n(s+k),p(s)}\,,\qquad \Delta^{(0)}_k \;=\; \epsilon\left(k+\epsilon_0+\frac{1}{2}\right). \tag{A.38}$$

The closure relation $[R_m, R_n] = -iM_{mn}$ holds for all $\lambda^{(s)}_k$, while $[R_m, P_n] = i\eta_{mn}R$ requires conformality. For example, for $s = 0$ the contributions to $[R_m, P_n]|(k)\rangle_{p(k)}$ of type $\eta_{\{p_1|(m}|(k)\rangle_{n)|p(k-1)\}}$ cancel iff $M_0^2 = M_0^2(\text{conf})$, leaving $8i\frac{\lambda^{(0)}_{k+1}(\text{conf})}{k+1}\eta_{mn}|(k)\rangle_{p(k)}$ to be identified with the action of R. We note that the conformal representation matrices of $\mathfrak{g}$ simplify, viz.

$$\lambda^{(s)}_k|_{\text{conf}} = \frac{\epsilon}{8}k(k+2s)\,,\qquad\qquad \text{for } s\geqslant 1 \text{ in } D=4\,, \tag{A.39}$$

$$\lambda^{(0)}_k|_{\text{conf}} = \frac{\epsilon}{8}k\left(k+\epsilon_0-\frac{1}{2}\right),\qquad \lambda^{(\frac{1}{2})}_k|_{\text{conf}} \;=\; \frac{\epsilon}{8}k(k+\epsilon_0)\,,\quad \mu^{(\frac{1}{2})}_k|_{\text{conf}} \;=\; 0\,. \tag{A.40}$$

The conformal embedding of $\mathfrak{iso}(D;\mathbb{C}) = \text{span}_{\mathbb{C}}\{M_{mn}, \Pi_m\}$ reads

$$\Pi_m = \frac{1}{\sqrt{2}}(P_m + R_m)\,,\quad \Sigma_m = \frac{1}{\sqrt{2}}(P_m - R_m)\,,\quad [\Pi_m, \Sigma_n] = iM_{mn} + i\eta_{mn}R\,, \tag{A.41}$$

with representation matrices $\Pi|^{(s,s)}_{(s+k,s)}\rangle = 2\sqrt{2}|^{(s,s)}_{(s+k+1,s)}\rangle$ and $\Sigma|^{(s,s)}_{(s+k,s)}\rangle = 2\sqrt{2}\lambda^{(s)}_k\eta|^{(s,s)}_{(s+k-1,s)}\rangle$, leading to the identifications $\Pi_m = \frac{1}{\sqrt{2}}L^+_m$, $\Sigma_m = \frac{1}{\sqrt{2}}L^-_m$ and $E = R$ (whose hermiticity properties are discussed in [50], for example), and to the identification of the smallest Lorentz tensor with the lowest-weight state. Thus, the light-likeness condition $\Pi^m\Pi_m \approx 0$ can be written $\mathfrak{so}(D+2;\mathbb{C})$-covariantly as the hyperlight-likeness condition

$$V_{\underline{AB}} \equiv \frac{1}{2}M_{\{\underline{A}}{}^{\underline{C}}M_{\underline{B}\}\underline{C}} \;\approx\; 0\,. \tag{A.42}$$

B. Details of the Lorentz-covariant factorization of $\mathcal{I}[V]$

This appendix contains details of the procedure of factoring out the ideal $\mathcal{I}[V]$ defined in (2.6) from the enveloping algebra $\mathcal{U}[\mathfrak{g}]$. The constraints $V_{AB} \approx 0$ and $V_{ABCD} \approx 0$, defined by (2.7) and (2.7), decompose under $\mathfrak{m}$ into

$$V_{\sharp\sharp} = \frac{1}{2}(\sigma P^a\star P_a - \mu^2)\;\approx 0\,,\qquad\qquad V_{\sharp a} = \frac{1}{4}\{M_a{}^b, P_b\}_\star \;\approx\; 0\,, \tag{B.1}$$

$$V_{ab} = \frac{1}{2}(M_{(a}{}^c\star M_{b)c} - \sigma P_{(a}\star P_{b)} + \mu^2\eta_{ab})\;\approx 0\,, \tag{B.2}$$

$$V_{abcd} = M_{[ab}\star M_{cd]} \approx 0\,,\qquad\qquad V_{\sharp abc} = -P_{[a}\star M_{bc]} = 0\,, \tag{B.3}$$

with $\mu^2 \equiv -\frac{2C_2[\mathcal{S}]}{D+1}$. The $\mathfrak{g}$-irreducibility of $V_{AB} \approx 0$ implies that $V_{\sharp a} \approx 0$ and $V_{ab} \approx 0$ follow from $V_{\sharp\sharp} \approx 0$. Similarly, the constraint $V_{abcd} \approx 0$ follows from $V_{\sharp abc} \approx 0$. More explicitly, using that μ^2 is a commuting element one can show that $V_{\sharp\sharp} \approx 0$ implies that

$$P^a\star M_{ab} \approx M_{ba}\star P^a \approx i(\epsilon_0+1)P_b\;, \tag{B.4}$$

from which (B.1) follows immediately (alternatively one may compute $V_{\sharp a} = -\frac{i\sigma}{4}[P_a, P^b\star P_b]_\star \approx -\frac{i}{4}[P_a,\mu^2]_\star = 0$). Next, eq. (B.4) and $P^a\star P_a \approx \sigma\mu^2$ imply

$$M_{(a}{}^c\star M_{b)c} \approx \sigma P_{(a}\star P_{b)} - \mu^2\eta_{ab}\,, \tag{B.5}$$

that is $V_{ab} \approx 0$. Similarly, $[P_{[a}, V_{0'bcd]}]_\star \propto V_{abcd}$, so that $V_{abcd} \approx 0$, i.e. $M_{[ab} \star M_{cd]} \approx 0$, follows from $V_{0'abc} \approx 0$, i.e. $P_{[a} \star P_b \star P_{c]} \approx 0$. Finally, the value of μ^2 is determined from $P^a \star P_{[a} \star P_b \star P_{c]} \approx \frac{i}{6}(\mu^2 - \epsilon_0) M_{bc}$.

Next, the $\mathfrak{m}$-covariant basis elements defined in (2.26) can be expanded in traces in the first row. For example, for $m = 0$ one has

$$T_{a(n)} = P_{\{a_1} \cdots P_{a_n\}} = P_{(a_1} \star \cdots \star P_{a_n)} + \kappa_{n,0;1} \eta_{(a_1 a_2} P_{a_3} \star \cdots \star P_{a_n)} + \mathcal{O}(\eta^2) \,, \tag{B.6}$$

with $\kappa_{n,0;1} = -\sigma \frac{(n+1)n(n-1)(n+4\epsilon_0-2)}{48(n+\epsilon_0-\frac{1}{2})}$. Let us use the contraction rules (2.24), (B.4) and (B.5) to compute $\kappa_n \equiv \kappa_{n,0;1}$ by demanding the right-hand side to be traceless. To this end, we first use (2.24) to expand

$$\begin{aligned} \eta^{bc} P_{(b} \star P_c \star P_{a_1} \cdots \star P_{a_{n-2})} &\approx \epsilon_0 P_{a_1} \cdots \star P_{a_{n-2}} \\ &+ \frac{2}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} P_{a_1} \cdots \star P_{a_{i-1}} \star [P_b, P_{a_i} \star \cdots \star P_{a_{j-2}}]_\star \star P^b \star P_{a_{j-1}} \star \cdots \star P_{a_{n-2}} \,, \end{aligned}$$

where the summand can be evaluated modulo trace parts, which only affect the higher traces in (B.6). Thus, for $i = 1$ and $j = n$ we find using (B.4) that $[P_b, P_{a_1} \star \cdots \star P_{a_{n-2}}]_\star \star P^b$

$$\begin{aligned} &= i M_{ba_1} \star P_{a_2} \star \cdots \star P_{a_{n-2}} \star P^b + P_{a_1} \star (i M_{ba_2}) \star \cdots \star P_{a_{n-2}} \star P^b + \cdots \\ &\approx (\epsilon_0 + 1) P_{a_1} \star \cdots \star P_{a_{n-2}} + \eta_{ba_2} P_{a_1} \star P_{a_3} \star \cdots \star P_{a_{n-2}} \star P^b \\ &\quad + P_{a_2} \star (\eta_{ba_3} P_{a_2}) \star P_{a_4} \star \cdots \star P_{a_{n-2}} \star P^b + P_{a_2} \star P_{a_3} \star (\eta_{ba_4} P_{a_3}) \star P_{a_5} \star \cdots \star P_{a_{n-2}} \star P^b + \cdots \\ &\quad + (\epsilon_0 + 1) P_{a_1} \star \cdots \star P_{a_{n-2}} + P_{a_2} \star (\eta_{ba_3} P_{a_2}) \star P_{a_4} \star \cdots \star P_{a_{n-2}} \star P^b + \cdots \\ &\quad + \cdots + \mathcal{O}(\eta) = \frac{1}{2}(n-2)(n + 2\epsilon_0 - 1) P_{a_1} \star \cdots \star P_{a_{n-2}} + \mathcal{O}(\eta) \,. \end{aligned}$$

The contributions from $j = n$ and $i = 1 + k$ for $k = 0, \ldots, n-2$ are obtained by letting $n \to n - k$, and are given modulo trace parts by $P_{a_1} \star \cdots \star P_{a_{n-2}}$ times $\sum_{k=0}^{n-2} k(\epsilon_0 + 1 + \frac{1}{2}(k-1)) = \frac{1}{6}(n-1)(n-2)(n+3\epsilon_0)$. Letting $n \to n-k$ yields the contributions from $j = n - k$ for $k = 0, \ldots, n-2$, which are thus given modulo trace parts by $P_{a_1} \star \cdots \star P_{a_{n-2}}$ times $\frac{1}{6} \sum_{k=0}^{n-2} (n-1-k)(n-2-k)(n-k+3\epsilon_0) = \frac{1}{24} n(n-1)(n-2)(n+4\epsilon_0+1)$. Hence the trace of the first term of the right-hand side of (B.6) is given by

$$\begin{aligned} \eta^{bc} P_{(b} \star P_c \star P_{a_1} \cdots \star P_{a_{n-2})} &\approx \left(\epsilon_0 + \frac{1}{12}(n-2)(n+4\epsilon_0+1) \right) P_{a_1} \star \cdots \star P_{a_{n-2}} + \mathcal{O}(\eta) \\ &= \frac{1}{12}(n+1)(n+4\epsilon_0-2) P_{a_1} \star \cdots \star P_{a_{n-2}} + \mathcal{O}(\eta) \,. \end{aligned} \tag{B.7}$$

Tracing the second term of the right-hand side of (B.6) yields κ_n times

$$\eta^{bc} \eta_{(bc} P_{a_1} \star \cdots \star P_{a_{n-2})} = \frac{4(n + 2\epsilon_0 - \frac{1}{2})}{n(n-1)} P_{a_1} \star \cdots \star P_{a_{n-2}} + \mathcal{O}(\eta) \,. \tag{B.8}$$

The tracelessness of the right-hand side of (B.6) thus requires

$$\frac{1}{12}(n+1)(n+4\epsilon_0-2) + \frac{2(2n+2\epsilon_0-1)}{n(n-1)} \kappa_n = 0 \,, \tag{B.9}$$

which yields the value of κ_n given below (B.6).

The contraction rules (2.24) and (B.4) can be verified against (A.26) which implies

$$\mathrm{ac}_{P_c}(T_{a(s+k),b(s)}) = 2\Delta_{s+k,s}T_{c\{a(s+k),b(s)\}} + 2\lambda^{(s)}_k \eta_{c\{a}T_{a(s+k-1),b(s)\}} , \tag{B.10}$$

where $\Delta_{s+k,s}$ and $\{\cdots\}$ are given below (A.26), and $\lambda^{(s)}_k$ in (A.28). For $s=0$ one has

$$\mathrm{ac}_{P_a}T_{b(n)} = \{P_a, T_{b(n)}\}_\star = 2T_{ab(n)} + 2\lambda^{(0)}_n \eta_{a\{b_1}T_{b(n-1)\}} , \tag{B.11}$$

with $\lambda^{(0)}_n = \frac{n(n+2\epsilon_0-1)(n+1)}{8(n+\epsilon_0+\frac{1}{2})}$ and

$$\eta_{a\{b_1}T_{b(n-1)\}} \equiv \eta_{a(b_1}T_{b(n-1))} + \alpha_n \eta_{(b_1b_2}T_{b(n-2))a} , \quad \alpha_n = \alpha_{n,0} = -\frac{n-1}{2(n+\epsilon_0-\frac{1}{2})} , \tag{B.12}$$

which means that $\lambda^{(0)}_n$ is determined by the trace condition

$$\{P^a, T_{ab(n-1)}\}_\star = \lambda^{(0)}_n \eta^{ac}\eta_{a\{b_1}T_{b(n-2)c\}} . \tag{B.13}$$

The right-hand side can been simplified using (B.12), which yields

$$\eta^{ac}\eta_{a\{b_1}T_{b(n-2)c\}} = \frac{(n+\epsilon_0+\frac{1}{2})(n+2\epsilon_0)}{n(n+\epsilon_0-\frac{1}{2})}T_{b(n-1)} . \tag{B.14}$$

On the left-hand side we first use the τ-map to show that $P^a \star T_{ab(n-1)} = T_{ab(n-1)} \star P^a$. We then use (B.6) and the contraction rules (2.24) and (B.4) to compute $P^a \star T_{ab(n-1)}$ as

$$\begin{aligned}
&\frac{1}{n}P^a \star \sum_{i=1}^{n} P_{b_1} \star \cdots P_{b_{i-1}} \star P_a \star P_{b_i} \star \cdots \star P_{b_{n-1}} + \frac{2\kappa_n}{n}P_{b_1} \star \cdots \star P_{b_{n-1}} + \mathcal{O}(\eta) \\
&\approx \left(\epsilon_0 + \frac{1}{2}(\epsilon_0+1)(n-1) + \frac{1}{6}(n-1)(n-2) + \frac{2\kappa_n}{n}\right) P_{b_1} \star \cdots \star P_{b_{n-1}} + \mathcal{O}(\eta) \\
&= \frac{(n+2\epsilon_0)(n+2\epsilon_0-1)(n+1)}{8(n+\epsilon_0-\frac{1}{2})}T_{b(n-1)} + \mathcal{O}(\eta) .
\end{aligned} \tag{B.15}$$

Substituting (B.14) and (B.15) into (B.13) then yields $\lambda^{(0}_n$ in agreement with (A.28).

Another basic $\star$-product in the $\mathfrak{m}$-covariant basis is

$$\mathrm{ac}_{M_{ab}}(T_{c(n)}) = 2T_{c(n)[a,b]} + 2\rho_n \eta_{[a|\{c_1}T_{c(n-1)\},|b]} , \quad \rho_n = -\frac{\sigma}{4}\frac{(n-1)n(n+1)}{n+\epsilon_0+\frac{1}{2}} , \tag{B.16}$$

with $\Delta_n = \frac{2(n+1)}{n+2}$, $\eta_{[a|\{c_1}T_{c(n-1)\},|b]} = \eta_{c_1[a|}T_{c(n-1),|b]} - \frac{n-1}{2(n+\epsilon_0-\frac{1}{2})}\eta_{c_1c_2}T_{c(n-2)[a,b]}$, and we note the alternative form $\mathrm{ac}_{M_{ab}}(T_{c(n)}) = 2M_{ab}T_{c(n)} + 2\rho_n\eta_{[a|\{c_1}T_{c(n-2)}M_{c_n\}|b]}$.

Finally, the Tr norms of the basis elements defined in (2.9) and (2.26) read

$$\mathrm{Tr}[M_{A(n),B(n)} \star M^{C(m),D(m)}] = \delta_{mn}\delta^{\{C(n),D(n)\}}_{\{A(n),B(n)\}}\mathcal{N}_{(n,n)_{D+1}} , \tag{B.17}$$

$$\mathrm{Tr}[T_{a(n),b(m)} \star T^{c(n'),d(m')}] = \delta_{n,n'}\delta_{m,m'}\delta^{\{c(n),d(m)\}}_{\{a(n),b(m)\}}\mathcal{N}_{(n,m)_D} , \tag{B.18}$$

where the normalizations are given by

$$\mathcal{N}_{(n,n)_{D+1}} = \lambda_n \lambda_{n-1} \cdots \lambda_1 = (-2)^{-n} \frac{n!(n+1)!(\epsilon_0)_n}{(\epsilon_0 + \frac{3}{2})_n}, \tag{B.19}$$

$$\mathcal{N}_{(n)_D} = \lambda_n^{(0)} \lambda_{n-1}^{(0)} \cdots \lambda_1^{(0)} = (8\sigma)^{-n} \frac{n!(n+1)!(2\epsilon_0)_n}{(\epsilon_0 + \frac{3}{2})_n}, \tag{B.20}$$

as can be seen by making repeated use of (2.12) and (A.26). For example,

$$\begin{aligned} \mathrm{Tr}[T_{a(n)} \star T_{b(n)}] &= \mathrm{Tr}[P_{\{a_1} \star \cdots \star P_{a_n\}} \star T_{b(n)}] \\ &= \mathrm{Tr}[P_{\{a_1} \star \cdots \star P_{a_{n-1}} \star \left(T_{n+1} + T_n + \lambda_n^{(0)} \eta_{a_n\}\{b_1} T_{b(n-1)\}}\right)] \\ &= \mathrm{Tr}[P_{\{a_1} \star \cdots \star P_{a_{n-1}} \star \lambda_n^{(0)} \eta_{a_n\}\{b_1} T_{b(n-1)\}}], \end{aligned} \tag{B.21}$$

where T_n denote traceless symmetric $\star$-products of n transvections, and we have used the fact that $\mathrm{Tr}[T_{n+1} \star T_{n-1}] = \mathrm{Tr}[T_n \star T_{n-1}] = 0$. In particular, using

$$\delta^{\{a(n)\}}_{\{b(n)\}} = \sum_{k=0}^{[n/2]} t_k (\eta^{a(2)} \eta_{b(2)})^k \delta^{a(n-2k)}_{b(n-2k)}, \qquad t_k = \frac{(-n)_{2k}}{4^k k!(-n-\epsilon_0+\frac{1}{2})_k}, \tag{B.22}$$

from which it follows that $\delta^{\{\sharp'(n)\}}_{\{\sharp'(n)\}} = \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{4^k k!(-n-\epsilon_0+\frac{1}{2})_k} = 2^{-n} \frac{(2\epsilon_0+1)_n}{(\epsilon_0+\frac{1}{2})_n}$, we obtain

$$\mathrm{Tr}[T_{\sharp'(n)} \star T_{\sharp'(n)}] = (-\sigma')^n 4^{-2n} \frac{n!(n+1)!(2\epsilon_0)_n (2\epsilon_0+1)_n}{(\epsilon_0+\frac{1}{2})_n (\epsilon_0+\frac{3}{2})_n}. \tag{B.23}$$

C. Quadratic and quartic Casimir operators

The values $C_{2n}[\mathfrak{g}|(e_0;\vec{s}_0)^\pm]$ of $C_{2n}[\mathfrak{g}] = \frac{1}{2} M_{A_1}{}^{A_2} \star M_{A_2}{}^{A_3} \star \cdots \star M_{A_{2n}}{}^{A_1}$ in $\mathfrak{D}^\pm(e_0;\vec{s}_0)$, $\vec{s}_0 = (m_1,\ldots,m_{\nu-1})$, are given for $n=1$ and $n=2$ by

$$C_2[\mathfrak{g}|(e_0;\vec{s}_0)^\pm] = x_0^\pm + C_2[\mathfrak{s}|\vec{s}_0], \tag{C.1}$$

$$C_4[\mathfrak{g}|(e_0;\vec{s}_0)^\pm] = x_0^\pm \left(x_0^\pm + \Delta_0\right) + C_4[\mathfrak{s}|\vec{s}_0] - C_2[\mathfrak{s}|\vec{s}_0], \tag{C.2}$$

where $x_0^\pm = e_0(e_0 \mp (D-1))$ and $\Delta_0 = \frac{1}{2}(D-1)(D-2)$, and the values of $C_2[\mathfrak{s}] = \frac{1}{2} M^{rs} \star M_{rs}$ and $C_4[\mathfrak{s}] = \frac{1}{2} M_r{}^s \star M_s{}^t \star M_t{}^u \star M_u{}^r$ in the $\vec{s}_0$-plet are given by

$$C_2[\mathfrak{s}|\vec{s}_0] = \sum_{k=1}^{\nu-1} x_k, \qquad C_4[\mathfrak{s}|\vec{s}_0] = \sum_{k=1}^{\nu-1} x_k (x_k + \Delta_k), \tag{C.3}$$

with $x_k = m_k(m_k + D - 1 - 2k)$ and $\Delta_k = \frac{1}{2}(D-1-2k)(D-2-2k) + 1 - k$. The ℓth level adjoint level $\mathcal{L}_\ell$ defined in (2.28) has lowest weight $(-(2\ell+1); 2\ell+1)$, which yields (2.30) and (2.31) (with $s = 2\ell+2$). The composite massless lowest-weight and highest-weight spaces $\mathfrak{D}^\pm(\pm(s+2\epsilon_0);(s))$ have the same values of $C_2[\mathfrak{g}]$ and $C_4[\mathfrak{g}]$.

To compute $C_2[\mathfrak{g}]$ in the twisted-adjoint representation we use $P^a \star P_a = \sigma\epsilon_0$ to write

$$\begin{aligned} \widetilde{\mathrm{ad}}_{C_2[\mathfrak{g}]}(S) &= \mathrm{ad}_{C_2[\mathfrak{m}]}(S) - \sigma\{P^a, \{P_a, S\}_\star\}_\star \\ &= \mathrm{ad}_{C_2[\mathfrak{m}]}(S) - 2\sigma(\epsilon_0 S + P^a \star S \star P_a) = \mathrm{ad}_{C_2[\mathfrak{g}]}(S_\ell) + 4\sigma P^a \star S \star P_a, \end{aligned} \tag{C.4}$$

where $C_2[\mathfrak{m}]=\frac{1}{2}M_{ab}M^{ab}$, from which $\sigma P^a\star S\star P_a$ can be eliminated, which yields

$$\widetilde{\mathrm{ad}}_{C_2[\mathfrak{g}]}(S) = 2\mathrm{ad}_{C_2[\mathfrak{m}]}(S)-\mathrm{ad}_{C_2[\mathfrak{g}]}(S_\ell)-4\epsilon_0 S\ . \tag{C.5}$$

An element $S_\ell\in\mathcal{T}_\ell$, defined by (2.29), carries the highest weights $(s+k,s)$ and $(s+k,s+k)$ with $s=2\ell+2$ and $k=0,1,\dots$ of the adjoint $\mathfrak{m}$ and $\mathfrak{g}$ actions, respectively, and for all k

$$\begin{aligned}\widetilde{\mathrm{ad}}_{C_2[\mathfrak{g}]}(S_\ell) &= \left(2C_2[\mathfrak{m}|(s+k,s)]-C_2[\mathfrak{g}|(s+k,s+k)]-4\epsilon_0\right)S_\ell\\ &= \left(2C_2[\mathfrak{m}|(s,s)]-C_2[\mathfrak{g}|(s,s)]-4\epsilon_0\right)S_\ell \ =\ C_2[\mathfrak{g}|\ell]S_\ell\ .\end{aligned}\tag{C.6}$$

Similarly,

$$\begin{aligned}\widetilde{\mathrm{ad}}_{C_4[\mathfrak{g}]}(S) &= \mathrm{ad}_{C_4[\mathfrak{m}]}(S)+\\ &\quad+\frac{\sigma}{2}[M_a{}^b,[M_b{}^c,\{P_c,\{P^a,S\}_\star\}_\star]_\star]_\star+\frac{\sigma}{2}[M_a{}^b,\{P_b,\{P^c,[M_c{}^a,S]_\star\}_\star\}_\star]_\star\\ &\quad+\frac{\sigma}{2}\{P_a,\{P^b,[M_b{}^c,[M_c{}^a,S]_\star]_\star\}_\star\}_\star+\frac{\sigma}{2}\{P^a,[M_a{}^b,[M_b{}^c,\{P_c,S\}_\star]_\star]_\star\}_\star\\ &\quad+\frac{1}{2}\{P_a,\{P^b,\{P_b,\{P^a,S\}_\star\}_\star\}_\star\}_\star+\frac{1}{2}\{P^a,\{P_b,\{P^b,\{P_b,S\}_\star\}_\star\}_\star\}_\star\\ &= \mathrm{ad}_{C_4[\mathfrak{m}]}(S)+C_+(S)+C_-(S)\ =\ \mathrm{ad}_{C_4[\mathfrak{g}]}(S)+2C_-(S)\,,\end{aligned}\tag{C.7}$$

where $C_+(S)$ and $C_-(S)$ are the terms with an even and odd number of translation generators standing to the right of S, respectively. Eliminating $C_-(S)$ leads to

$$\widetilde{\mathrm{ad}}_{C_4[\mathfrak{g}]}(S) = 2\mathrm{ad}_{C_4[\mathfrak{m}]}(S)-\mathrm{ad}_{C_4[\mathfrak{g}]}(S)+2C_+(S)\ . \tag{C.8}$$

The quantity $C_+(S)$ can be calculated using (B.4), (B.5), and $M_{ab}\star S\star M^{ab}=-\mathrm{ad}_{C_2[\mathfrak{m}]}(S)+\{M^{ab}\star M_{ab},S\}_\star$, with the result

$$C_+(S) = \mathrm{ad}_{C_2[\mathfrak{m}]}(S)-2\epsilon_0(2\epsilon_0^2-\epsilon_0+1)S\ . \tag{C.9}$$

Thus, using the $\mathfrak{g}$-adjoint and $\mathfrak{m}$-adjoint highest weights for S_ℓ, we find that

$$\begin{aligned}\widetilde{\mathrm{ad}}_{C_4[\mathfrak{g}]}(S_\ell) &= \left(C_4[\mathfrak{m}|(s+k,s)]-C_4[\mathfrak{g}|(s+k,s+k)]\right)S_\ell\\ &\quad+\big(C_2[\mathfrak{m}|(s+k,s)]-4\epsilon_0(2\epsilon_0^2-\epsilon_0+1)\big)S_\ell \ =\ C_4[\mathfrak{g}|\ell]S_\ell\ .\end{aligned}\tag{C.10}$$

D. Critical and conformal masses for the Weyl zero-forms

In this appendix we give some details related to (2.70) and the mass formula (2.74). Let us begin with the case of $s=0$, where the linearized master-field constraint reads $\nabla\Phi_{(0)}-ie^a\{P_a,\Phi_{(0)}\}_\star=0$, with $\Phi_{(0)}$ given by (2.64) and P_a in the presentation (B.11). The constraint takes the component form

$$\nabla_b\Phi_{a(n)}-2n\eta_{b\{a_1}\Phi_{a(n-1)\}}+\frac{2\lambda_n^{(0)}}{n+1}\Phi_{ba(n)} = 0\,, \tag{D.1}$$

with $\eta_{b\{a_1}\Phi_{a(n-1)\}}$ given by (B.12). The symmetric and traceless part of (D.1) yields (2.71) for $s=0$ while its trace part leads to the masses in (2.74). To this end, contraction with ∇^b yields

$$\nabla^2\Phi_{a(n)}-2n\eta^{bc}\left(\eta_{ba_1}\nabla_c\Phi_{a(n-1)}+\alpha_n\eta_{a_1a_2}\nabla_c\Phi_{a(n-2)b}\right)+\frac{2\lambda_{n+1}^{(0)}}{n+1}\nabla^b\Phi_{ba(n)} = 0\ . \tag{D.2}$$

Elimination of $\nabla_c\Phi_{a(n-1)}$ and $\nabla_c\Phi_{a(n1)}$ using (2.71) followed by $\{a(n)\}$-projection leads to

$$\nabla^2\Phi_{a(n)} + 4\lambda_n^{(0)}\Phi_{a(n)} + 4\lambda_{n+1}^{(0)}\eta^{bc}\left(\eta_{c(b}\Phi_{a(n))} + \alpha_{n+1}\eta_{(ba_1}\Phi_{a(n-1))c}\right) = 0\ . \tag{D.3}$$

Performing the traces one ends up with the following expression for the critical mass:

$$M^2_{0,n} = -4\lambda_n^{(0)} - 4\lambda_{n+1}^{(0)}\frac{(n+2\epsilon_0+1)(n+\epsilon_0+\frac{3}{2})}{(n+1)(n+\epsilon_0+\frac{1}{2})}\ . \tag{D.4}$$

Inserting (A.28) leads to the critical mass $M^2_{0,n} = -(n^2+(2\epsilon_0+1)n+4\epsilon_0)\sigma$ in agreement with (2.74). For general s we use (A.26) and (2.64) to expand $\nabla_c\Phi - i\{P_c,\Phi\}_\star$ as

$$\sum_{s,n}\frac{i^n}{n!}\Big(T_{a(s+n),b(s)}\nabla_c - 2i(\Delta_{n+s,s}T_{ca(s+n),b(s)} + \lambda_n^{(s)}\eta_{c\{a}T_{a(s+n-1),b(s)\}})\Big)\Phi^{a(s+n),b(s)}\ . \tag{D.5}$$

The component form (2.70) follows by rewriting the middle term as $T_{ca(s+n),b(s)}\Phi^{a(s+n),b(s)} = T^{a(s+n+1),b(s)}\eta_{c\{a}\Phi_{a(s+n),b(s)\}}$ and last term as

$$\begin{aligned}\eta_{c\{a}T_{a(s+n-1),b(s)\}}\Phi^{a(s+n),b(s)} &= \big(\eta_{ca}T_{a(s+n-1),b(s)} + (\eta_{aa}\text{ and }\eta_{ab}\text{ traces})\big)\,\Phi^{a(s+n),b(s)}\\ &= T^{a(s+n-1),b(s)}\Phi_{c\{a(s+n-1),b(s)\}}\ .\end{aligned} \tag{D.6}$$

Contracting (2.70) by ∇^c yields

$$\nabla^2\Phi_{a(s+n),b(s)} = 2n\Delta_{s+n-1,s}\eta^{cd}\nabla_d\eta_{c\{a}\Phi_{a(s+n-1),b(s)\}} - \frac{2\lambda^{(s)}_{n+1}}{n+1}\eta^{cd}\nabla_d\Phi_{c\{a(s+n),b(s)\}}\ , \tag{D.7}$$

with $\eta_{c\{a}\Phi_{a(s+n-1),b(s)\}}$ given by (A.27), and where the gradients on the right-hand side are to be eliminated using (2.70). In the first term one finds

$$\begin{aligned}&-4\Delta_{s+n-1,s}\lambda_n^{(s)}\eta^{cd}\big(\eta_{ca}\Phi_{d\langle a(s+n-1),b(s)\rangle} + \alpha_{s+n,s}\eta_{a(2)}\Phi_{d\langle a(s+n-2)c,b(s)\rangle}\\ &+\beta_{s+n,s}\eta_{a(2)}\Phi_{d\langle a(n+s-2)b,cb(s-1)\rangle} + \gamma_{s+n,s}\eta_{ab}\Phi_{d\langle a(s+n-1),cb(s-1)\rangle}\big)\\ &\qquad = -4\Delta_{s+n-1,s}\lambda_n^{(s)}\Phi_{a\langle a(s+n-1),b(s)\rangle} = -4\lambda_n^{(s)}\Phi_{a(s+n),b(s)}\ ,\end{aligned} \tag{D.8}$$

where the $\langle\cdots\rangle$ Young projections are imposed *prior* to the final symmetrization on a and b indices and $\Phi_{a\langle a(s+n-1),b(s)\rangle} = (\Delta_{s+n-1,s})^{-1}\Phi_{a(s+n),b(s)}$, which follows from the definition of $\Delta_{s+n,s}$ in (A.26). In the second term

$$\begin{aligned}&-4\lambda^{(s)}_{n+1}\Delta_{s+n,s}\eta^{cd}\eta_{c\{d}\Phi_{\{a(s+n),b(s)\}\}}\\ &= -4\lambda^{(s)}_{n+1}\Delta_{s+n,s}\frac{1}{n+s+1}\left(s+n+2\epsilon_0+3+2\alpha_{s+n+1,s} - \frac{2\beta_{s+n+1,s}}{s+n} + \gamma_{s+n+1,s}\right)\Phi_{a(s+n),b(s)}\\ &= -4\lambda^{(s)}_{n+1}\Delta_{s+n,s}\frac{(n+s+2\epsilon_0)(n+s+\epsilon_0+\frac{3}{2})(n+2s+2\epsilon_0+1)}{(n+s+1)(n+2s+2\epsilon_0)(n+s+\epsilon_0+\frac{1}{2})}\Phi_{a(s+n),b(s)}\ .\end{aligned} \tag{D.9}$$

Combining (D.8) and (D.9) the critical mass can be identified with (2.74).

Unfolding a scalar field ϕ obeying $(\nabla^2 - M_0^2)\phi = 0$ yields the master-field equation $(\nabla - ie^aP_a)|\Phi_{(0)}\rangle = 0$ where $|\Phi_{(0)}\rangle = \sum_{n=0}^{\infty}\frac{i^n}{n!}\Phi^{a(n)}|(n)\rangle_{a(n)}$ belongs to $\mathfrak{R}(0)$ defined by (A.26) with $\epsilon = \sigma$ and $\mathfrak{m}_1 = \mathfrak{m}$. From $[\nabla_a, \nabla_b]V_c = 2\sigma\eta_{c[b}V_{a]}$ it follows that the auxiliary 0-forms obey

$$(\nabla^2 - M_{0,n}^2)\Phi_{a(n)} = 0, \qquad M_{0,n}^2 = \sigma(M_0^2 - (n + 2\epsilon_0 + 1)n), \tag{D.10}$$

for non-critical masses obeying (D.4) with $\lambda_n^{(s)}$ given by (A.29). Similarly, unfolding a spinor ψ obeying $(\gamma^a\nabla_a + \epsilon' M_{1/2})\psi = 0$ yields the master-field equation $(\nabla - ie^a P_a)|\Psi_{(1/2)}\rangle = 0$ where $|\Psi_{(1/2)}\rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!}\overline{\Psi}^{a(n)}|(n+\frac{1}{2})\rangle_{a(n)}$ belongs to $\mathfrak{C}(\frac{1}{2})$ defined by (A.33) with $\epsilon = \sigma$ and ϵ' is the sign in $\overline{\psi}^{\beta}(\gamma_a)_{\beta}{}^{\alpha} = \epsilon'(\gamma_a)^{\alpha\beta}\psi_\beta$. The conformal masses in the maximally symmetric D-dimensional geometry with $R_{ab,cd} = -2\sigma\eta_{a[c}\eta_{d]b}$ are given by

$$M_0^2|_{\rm conf} = \frac{(D-2)\eta^{ac}\eta^{bd}R_{ab,cd}}{4(D-1)} = -\sigma\left(\epsilon_0 + \frac{1}{2}\right)\left(\epsilon_0 + \frac{3}{2}\right), \quad M_{1/2}^2|_{\rm conf} = 0, \tag{D.11}$$

We note that $M_0^2|_{\rm conf} = M_0^2|_{\rm crit}$ iff $D = 4$ or $D = 6$ and that $M_{1/2}^2|_{\rm conf} = M_{1/2}^2|_{\rm crit}$ iff $D = 4$.

E. Indecomposable negative-spin extension of scalar singletons

The scalar singleton $\mathfrak{D}_0^+ = \mathfrak{D}^+(\epsilon_0; (0))$, which consists of compact $(\mathfrak{e} \oplus \mathfrak{s})$ weights $(\epsilon_0 + j; (j))$ with $j = 0, 1, \ldots$, can be embedded together with the scalar anti-singleton $\mathfrak{D}_0^- = \mathfrak{D}^-(-\epsilon_0; (0))$ into the indecomposable representation

$$\mathfrak{M}_0 = \mathfrak{W}_0 \supsetplus (\mathfrak{D}_0^+ \oplus \mathfrak{D}_0^-), \qquad \mathfrak{W}_0 = \bigoplus_{(e;(\nu)) \in \Lambda(\epsilon_0)} \mathfrak{W}_{e;(\nu)}, \tag{E.1}$$

where $\Lambda(\epsilon_0) = \{(\pm([\epsilon_0] + p); (-(\epsilon_0 + [\epsilon_0] + p)))\}_{p=0}^{\infty}$ and $\text{mult}(e; (\nu)) = 1$. The negative spins (ν) label $\mathfrak{s}$-irreps $\mathfrak{W}_{e;(\nu)}|_{\mathfrak{s}} \simeq \mathfrak{W}(\nu)$ given by the quotient submodule sitting in the dual $\widetilde{\mathfrak{R}}_{\mathfrak{s}}(0)$ of the twisted-adjoint $\mathfrak{s}$-module $\mathfrak{R}_{\mathfrak{s}}(0)$ defined by (A.25), (A.26) and (A.28) with $D \to D - 2$ and $C_2[\mathfrak{s}|(0)] = \nu(\nu + 2\epsilon_0) = p^2 - \epsilon_0^2$. This twisted-adjoint module is irreducible for $p \leqslant \epsilon_0 - [\epsilon_0] - 1$ and indecomposable for $p \geqslant \epsilon_0 - [\epsilon_0]$, in which case its dual $\widetilde{\mathfrak{R}}_{\mathfrak{s}}(0)$ contains $\mathfrak{D}(j)$ with $j = p - \epsilon_0 + [\epsilon_0]$ as an invariant subspace. The representation matrix of $\widetilde{\mathfrak{R}}_{\mathfrak{s}}(0)$ is given below in (E.8) and (E.9). The decomposition $\mathfrak{M}_0|_{\mathfrak{g}'} = \mathcal{M}_{(0)}^{(+)} \oplus \mathcal{M}_{(0)}^{(-)}$ under $\mathfrak{g}' = \mathfrak{so}(D-2, 2) \subset \mathfrak{g}$ is the twisted-adjoint compact-weight space of the harmonic expansion of a conformal scalar field[37] in AdS_{D-1}, such that $\mathfrak{W}_0|_{\mathfrak{g}'} = \mathcal{W}_{(0)}^{(+)} \oplus \mathcal{W}_{(0)}^{(-)}$ and $\mathfrak{D}_0^{\pm}|_{\mathfrak{g}'} = \mathfrak{D}^{\pm}(\pm\epsilon_0; (0)) \oplus \mathfrak{D}^{\pm}(\pm(\epsilon_0 + 1); (0))$, with

$$\mathcal{M}_{(0)}^{(\pm)} = \bigoplus_{\substack{e \in \mathbb{Z} + [\epsilon_0],\ j' \in \{0,1,\ldots\} \\ e + j' = [\epsilon_0] + \frac{1 \mp 1}{2} \bmod 2}} \mathbb{C} \otimes |e; (j')\rangle = \mathcal{W}_{(0)}^{(\pm)} \supsetplus \left[(1 + \pi)\mathfrak{D}^+(\epsilon_0 + \delta_\pm; (0))\right], \tag{E.2}$$

where $\delta_\pm = \frac{1}{2}(1 \mp (-1)^{\epsilon_0 - [\epsilon_0]})$. We note that the action of $\mathfrak{g}'$ is unitarizable in $\mathfrak{M}_0$, while the action of $\mathfrak{g}$ is only unitarizable in $\mathfrak{D}_0^{\pm}$.

[37] Alternatively, on $S^1 \times S^{D-2}$ with metric $ds^2 = -dt^2 + d\Omega_{S^{D-2}}^2$, the harmonic expansion reads $\phi = \sum_{\omega,\nu,\alpha} \phi_{\omega,\nu,\alpha} e^{i\omega t} D_{\nu,\alpha}(\widehat{n})$ where $|\omega| = -\nu - \epsilon_0$ and $(\nabla_{S^{D-2}}^2 + \nu(\nu + 2\epsilon_0))D_{\nu,\alpha} = 0$ for $\nu = -(\epsilon_0 + [\epsilon_0] + p)$ with $p = 0, 1, \ldots$. For fixed $\nu \equiv -2\epsilon_0 - \ell$, the generalized spherical harmonics $\{D_{\nu,\alpha}\} \simeq \mathfrak{W}(\nu)$, which is irreducible if $\ell < 0$ and indecomposable if $\ell \geqslant 0$ with ideal consisting of the standard spherical harmonics $Y_{\ell,\alpha}$.

Explicitly, letting $\mathfrak{m}' = \mathfrak{so}(D-2,1) \subset \mathfrak{so}(D-2,2)$, the harmonic map from the conformal twisted-adjoint $(\mathfrak{g}'|\mathfrak{m}')$ module $\mathfrak{R}(0)|_{\mathfrak{g}'}$ (given by (A.26), (A.37), (A.38) and (A.40) for $D \to D-1$) to $\mathcal{M}_0$ takes the form

$$|^{(0)\mathrm{c}}_{e;(j')}\rangle_{r'(j')} = \sum_{n=0}^{\infty} f^{(0)\mathrm{c}}_{e;(j');n} |^{(0)}_{(n+j')}\rangle_{0(n)\{r'(j')\}}\,, \tag{E.3}$$

where $|^{(0)}_{(n)}\rangle_{a'(n)} \in \mathfrak{R}(0)|_{\mathfrak{g}'}$ $(a' = 0,1,\ldots,D-2)$ and $|^{(0)\mathrm{c}}_{e;(j')}\rangle_{r'(j')}$ $(r' = 1,\ldots,D-2)$ is a type-(j') tensor of $\mathfrak{s}' = \mathfrak{so}(D-2)$ with energy $(E-e)|^{(0)\mathrm{c}}_{e;(j')}\rangle_{r'(j')} = 0$. The latter condition yields $\theta(n-1) f^{(0)\mathrm{c}}_{e;(j');n-1} - \frac{e}{2} f^{(0)\mathrm{c}}_{e;(j');n} - \frac{1}{16}(n+1)(n+2(j'+\epsilon_0)) f^{(0)\mathrm{c}}_{e;(j');n+1} = 0$ for $n \geqslant 0$, where $\theta(x)$ equals 1 for $x \geqslant 0$ and 0 for $x < 0$. Equivalently, the generating function $f^{(0)\mathrm{c}}_{e;(j')}(z) = \sum_{n=0}^{\infty} z^n f^{(0)\mathrm{c}}_{e;(j');n}$ obeys

$$\left(\frac{z}{16}\frac{d^2}{dz^2} + \frac{j'+\epsilon_0}{8}\frac{d}{dz} + \frac{e}{2} - z\right) f^{(0)\mathrm{c}}_{e;(j')}(z) = 0\,, \tag{E.4}$$

whose solutions that are analytic at $z = 0$ can be written as the closed contour integrals[38]

$$f^{(0)\mathrm{c}}_{e;(j')}(z) = \mathcal{C}^{(0)\mathrm{c}}_{e;(j')} \oint_C \frac{ds}{2\pi i} \delta_{e,j'}(s)(1-s)^{\epsilon_0+j'+e-1}(1+s)^{\epsilon_0+j'-e-1} e^{4sz}\,, \tag{E.7}$$

where $\mathcal{C}^{(0)\mathrm{c}}_{e;(j')}$ is a normalization constant chosen such that $f^{(0)\mathrm{c}}_{e;(j')}(0) = 1$; $\delta_{e,j'}(s) = 1$ for $|e| \geqslant \epsilon_0 + j'$ and $\delta_{e'j'}(s) = \log\frac{s+1}{s-1}$ for $|e| \leqslant \epsilon_0 + j' - 1$; and C is a closed contour encircling the branch cut from $[-1,1]$. The integral collapses on residues at $s = \pm 1$ for $|e| \geqslant \epsilon_0 + j'$, i.e. for elements in the lowest-weight and highest-weight spaces, while it collapses on the (logarithmic) branch cut and turns into a real line integral from $s = -1$ to $s = +1$ for $|e| \leqslant \epsilon_0 + j' - 1$, i.e. for elements in the lowest-spin spaces.

The non-compact $\mathfrak{s}$ module $\mathfrak{W}(\nu)$ thus consists of the states $|^{(0)\mathrm{c}}_{e;(j')}\rangle$ on which the $\mathfrak{s}$ action is represented by $M_{r's'}$, generating the $\mathfrak{s}'$ subalgebra of $\mathfrak{s}$, and the conformal translations

$$\frac{1}{2} R_{r'} |^{(0)\mathrm{c}}_{e;(j')}\rangle_{s'(j')} = \widetilde{\rho}^{(e)}_{j'} |^{(0)\mathrm{c}}_{e;(j'+1)}\rangle_{r's'(j')} - \widetilde{\lambda}^{(e)}_{j'} \delta_{r'\{} |^{(0)\mathrm{c}}_{e;(j'-1)}\rangle_{s'(j'-1)\}}\,, \tag{E.8}$$

$$\widetilde{\rho}^{(e)}_{j'} = \frac{(j'+e+\epsilon_0)(j'-e+\epsilon_0)}{(j'+\epsilon_0)(j'+\epsilon_0+\frac{1}{2})}\,, \qquad \widetilde{\lambda}^{(e)}_{j'} = \frac{1}{8} j'(j'+\epsilon_0-1)\,, \tag{E.9}$$

[38]The line integral $\int_a^b ds f(s)$, where $a, b \in \mathbb{R}$ and $f(s)$ is analytic in a neighborhood of $[a,b]$ can be rewritten as the closed contour integral

$$\int_a^b ds f(s) = \oint_\gamma \frac{ds}{2\pi i} \log\left(\frac{s-a}{s-b}\right) f(s)\,, \tag{E.5}$$

where γ encircles $[a,b]$. For example, if $n = 0,1,\ldots$ then

$$\int_0^1 ds s^n = \oint_\gamma \frac{ds}{2\pi i} \log\left(\frac{s}{s-1}\right) s^n = \oint_0 \frac{ds}{2\pi i} \log\left(\frac{1}{1-s}\right) s^{-n-2} = \frac{1}{n+1}\,. \tag{E.6}$$

This lemma generalizes to the case where $f(s)$ has branch-cut along $[a,b]$ provided $f(s) \sim (s-s_0)^{\eta_{s_0}}$ with $\eta_{s_0} > -1$ as $s \sim s_0 \in \{a,b\}$, so that the real line integral is finite.

JHEP10(2008)022

as can be seen by acting on (E.3) with $R_{r'}$ and using (A.37); one first obtains $\frac{1}{2}R_{r'}|^{(0)c}_{(n+j')}\rangle_{s'(j')0(n)} = |^{(0)c}_{(n+j'+1)}\rangle_{s'(j')0(n)} - \frac{1}{8}(n+j')(n+j'+\epsilon_0-1)\eta_{r'\{s'_1}|^{(0)c}_{(n+j'-1)}\rangle_{s'(j')0(n)\}_{D-1}}$, which yields

$$\frac{1}{2}R_{r'}|^{(0)c}_{(n+j')}\rangle_{\{s'(j')\}0(n)} = \sum_{p=0,2}\Big(\widetilde{\rho}^{\,n,j'}_p|^{(0)c}_{(n+j'-p)}\rangle_{\{r's'(j')\}0(n-p)} \\ -\widetilde{\lambda}^{n,j'}_p\delta_{r'\{s'_1}|^{(0)c}_{(n+j'+p)}\rangle_{s'(j'-1)\}0(n+p)}\Big)$$

with $\widetilde{\rho}^{\,n,j'}_0 = 1$, $\widetilde{\rho}^{\,n,j'}_2 = -\frac{1}{16}n(n-1)$, $\widetilde{\lambda}^{n,j'}_0 = \frac{1}{8}j'(n+j'+\epsilon_0-1) + \frac{1}{32}\frac{n(n-1)j'}{j'+\epsilon_0-\frac{1}{2}}$ and $\widetilde{\lambda}^{n,j'}_2 = -\frac{1}{2}\frac{j'}{j'+\epsilon_0-\frac{1}{2}}$, and where $\{\cdots\}$ denotes the traceless type-(j') projection. The coefficients in (E.9) are then given by $\widetilde{\rho}^{\,(e)}_{j'} = \widetilde{\rho}^{\,0,j'}_0 + \widetilde{\rho}^{\,2,j'}_2 f^{(0)c}_{e;(j');2}$ and $\widetilde{\lambda}^{(e)}_{j'} = \widetilde{\lambda}^{0,j'}_0$.

The Flato-Fronsdal construction raises the issue of what is the additional content of the direct product $\mathfrak{M}_0 \otimes \mathfrak{M}_0$. It cannot be identified with $\mathcal{M}_{(0)}$, which has already the factorization given in (3.32) in terms of two angletons. Indeed, in odd dimensions the shadow $|2;(0)\rangle_{12}$, which obey $(E(\xi)-1)|2;(0)\rangle_{12} = 0$ and $L^-_r(\xi)|2;(0)\rangle_{12} = 0$ $(\xi = 1,2)$, cannot be realized in $\mathfrak{M}_0 \otimes \mathfrak{M}_0$ except in the trivial case of $D = 5$.

F. Some properties of the $T^{(0)}_{e;(j)}$ elements

To analyze the elements $T^{(0)}_{e;(j)}$ defined by (3.3), we start from (A.26) which for $\sigma = 1$ implies

$$\widetilde{P}_0 T_{0(n)\{r(j)\}} = 2\left(T_{0(n+1)\{r(j)\}} - \frac{1}{16}\lambda_{(j);n}T_{0(n-1)\{r(j)\}}\right), \tag{F.1}$$

$$\lambda_{(j);n} = \frac{n(n+j+1)(n+j+2\epsilon_0-1)(n+2j+2\epsilon_0)}{(n+j+\epsilon_0+\frac{1}{2})(n+j+\epsilon_0-\frac{1}{2})}. \tag{F.2}$$

Thus, the coefficients $f^{(0)}_{e;(j);n}$ obey the recursion relation

$$\theta(n-1)f^{(0)}_{e;(j);n-1} - \frac{e}{2}f^{(0)}_{e;(j);n} - \frac{1}{16}\lambda_{(j);n+1}f^{(0)}_{e;(j);n+1} = 0, \tag{F.3}$$

with initial condition $f^{(0)}_{e;(0);0} = 1$ and . If $j = 0$ one has

$$E \star E^n = E^{n+1} - \frac{1}{16}\lambda_{(0);n}E^{n-1}, \qquad \lambda_{(0);n} = \frac{n(n+1)(n+2\epsilon_0-1)(n+2\epsilon_0)}{(n+\epsilon_0-\frac{1}{2})(n+\epsilon_0+\frac{1}{2})}, \tag{F.4}$$

where $E^n \equiv T_{0(n)}$, and the recursion relation can be rewritten as

$$2e\left(n+\epsilon_0+\frac{1}{2}\right)\widetilde{f}_n = (n+1)(n+2\epsilon_0)\left(\widetilde{f}_{n-1} - \widetilde{f}_{n+1}\right), \quad f_n = \frac{4^n(\epsilon_0+\frac{3}{2})_n}{(n+1)!(2\epsilon_0+1)_n}\widetilde{f}_n. \tag{F.5}$$

If $e = 0$ then $f^{(0)}_{0;(0);2p+1} = 0$ and

$$f^{(0)}_{0;(0);2p} = 4^{2p}\frac{(\epsilon_0+\frac{3}{2})_{2p}}{(2)_{2p}(2\epsilon_0+1)_{2p}} = 2^{2p}\frac{\left(\frac{2\epsilon_0+5}{4}\right)_p\left(\frac{2\epsilon_0+3}{4}\right)_p}{p!\left(\frac{3}{2}\right)_p(\epsilon_0+1)_p\left(\epsilon_0+\frac{1}{2}\right)_p}, \tag{F.6}$$

and one may write the generating function as

$$f^{(0)}_{0;(0)}(z) = \frac{1}{2E}\int_0^E dz\left({}_1F_1\left(\epsilon_0+\frac{3}{2};2\epsilon_0+1;4z\right)+{}_1F_1\left(\epsilon_0+\frac{3}{2};2\epsilon_0+1;-4z\right)\right)$$
$$= {}_2F_3\left(\frac{2\epsilon_0+3}{4},\frac{2\epsilon_0+5}{4};\frac{3}{2},\epsilon_0+\frac{1}{2},\epsilon_0+1;4z^2\right)\ . \tag{F.7}$$

Since $T^{(0)}_{e;(0)}$ obeys $(E-\frac{e}{2})\star T^{(0)}_{e;(0)}=0$ it can be represented as

$$T^{(0)}_{e;(0)} = \mathcal{C}^{(0)}_{e;(0)}\oint_C \frac{d\alpha(s)}{2\pi i}\delta_e(s)\exp\left(-\frac{e\alpha(s)}{2}\right)g(s;E)\,,\qquad g(s;E)\ =\ \exp_\star[\alpha(s)E]\ , \tag{F.8}$$

where $g(s;E)$ is the Weyl-ordered form of the group element; C is a closed and bounded contour; $\delta_e(s)$ equals 1 if there are poles or branch cuts in the remaining part of the integrand, which is the case for generic e, and $\log[(s-a)/(s-b)]$ for suitable a and b for special values of e; and $\mathcal{C}^{(0)}_{e;(0)}$ a normalization chosen such that $f^{(0)}_{e;(0)}(0)=1$. In $D=4,6$ the composite massless scalars are conformal (see appendix D), with simpler representations matrices, viz. $\lambda^{(0)}_n = \frac{\sigma}{8}n(n+\epsilon_0-\frac{1}{2})$ and $\lambda_{(0);n}=n(n+2\epsilon_0)$, and the group element can be written as

$$g(s;E) = \frac{1}{\cosh^{2\epsilon_0+1}\frac{\alpha(s)}{2}}\exp\left[4E\tanh\frac{\alpha(s)}{4}\right]. \tag{F.9}$$

Specifically, in $D=4$, one has $E\star T(E) = \left(1-\frac{1}{16}\frac{d^2}{dE^2}\right)ET(E)$, implying $\left(\frac{1}{16}\frac{d^2}{dz^2}+\frac{e}{2z}-1\right)zf^{(0)}_{e;(0)}(z)=0$, and $f^{(0)}_{e;(0)}(z)=e^{-4z}{}_1F_1(1-e;2;8z)$ for $e\in\mathbb{C}$. These functions can be represented via the Laplace transformations

$$e\neq 0: \qquad f^{(0)}_{e;(0)}(z) = -\frac{1}{2e}\oint_\gamma\frac{ds}{2\pi i}\left(\frac{s-1}{s+1}\right)^e e^{4sz}\,, \tag{F.10}$$

$$e=0: \qquad f^{(0)}_{0;(0)}(z) = \frac{1}{2}\oint_\gamma\frac{ds}{2\pi i}\log\left(\frac{s+1}{s-1}\right)e^{4sz}\,, \tag{F.11}$$

where the closed contour γ encircles the interval $[-1,+1]$ which is a branch cut except for $e\in\{\pm1,\pm2,\dots\}$. In the latter case, the contour encloses the pole at $-\mathrm{sign}(e)$ with residue the rescaled Laguerre polynomial

$$e\in\{\pm1,\pm2,\dots\}:\ f^{(0)}_{e;(0)}(z)\ =\ \frac{e^{-4\mathrm{sign}(e)z}}{|e|}L^1_{|e|-1}(8\mathrm{sign}(e)z)\ . \tag{F.12}$$

The integral (F.10) approaches (F.11) as $e\to 0$, since $\left(\frac{s-1}{s+1}\right)^e = 1+e\log\frac{s-1}{s+1}+\mathcal{O}(e^2)$, and (F.11) can be rewritten as the real line integral

$$e=0:\ f^{(0)}_{0;(0)}(z)\ =\ \frac{1}{2}\int_{-1}^1 ds\,e^{4sz}\ =\ \frac{\sinh 4z}{4z}\ . \tag{F.13}$$

In $D=6$ the functions take on a similar form:

$$e\neq 0,\pm1: \quad f^{(0)}_{e;(0)}(z) = \frac{3}{4e(e^2-1)}\oint_\gamma\frac{ds}{2\pi i}(1-s^2)\left(\frac{s-1}{s+1}\right)^e e^{4sz}\,, \tag{F.14}$$

$$e = 0, \pm 1: \quad f^{(0)}_{0;(0)}(z) = \frac{3}{4(1+e^2)} \oint_\gamma \frac{ds}{2\pi i}(1-s^2)\left(\frac{1-s}{s+1}\right)^e \log\left(\frac{s+1}{s-1}\right) e^{4sz} \ . \qquad \text{(F.15)}$$

If $e = 2$ or $e = 2\epsilon_0$ it is possible to impose the lowest-weight condition

$$e = 2, 2\epsilon_0 \ : \ \widetilde{L}^-_r T^{(0)}_{e;(0)} \ = \ 0 \ , \qquad \text{(F.16)}$$

which by means of

$$\widetilde{L}^-_r T_{0(n)} \ = i\left(nT_{r0(n-1)} + 2T_{r0(n)} + \frac{1}{8}\lambda'_{(0);n} T_{r0(n-2)}\right) , \qquad \text{(F.17)}$$

$$\lambda'_{(0);n} \ = \ \frac{(n-1)n(n+1)(n+2\epsilon_0-1)}{(n+\epsilon_0-\frac{1}{2})(n+\epsilon_0+\frac{1}{2})} , \qquad \text{(F.18)}$$

amounts to the recursive relation $(n \geqslant 0)$

$$e = 2, 2\epsilon_0 \ : \ f^{(0)}_{e;(0);n} + \frac{n+1}{2} f^{(0)}_{e;(0);n+1} + \frac{1}{16}\lambda'_{(0);n+2} f^{(0)}_{e;(0);n+2} \ = \ 0 \ . \qquad \text{(F.19)}$$

Subtracting (F.3) yields the first-order difference equation $(n \geqslant 0)$

$$(n+1+e) f^{(0)}_{e;(0);n+1} + \frac{1}{8}(\lambda'_{(0);n+2} + \lambda_{(0);n+2}) f^{(0)}_{e;(0);n+2} \ = \ 0 , \qquad \text{(F.20)}$$

$$\lambda'_{(0);n+2} + \lambda_{(0);n+2} \ = \ 2\frac{(n+2)(n+3)(n+2\epsilon_0+1)}{n+\epsilon_0+\frac{5}{2}} , \qquad \text{(F.21)}$$

whose solution can be shown to obey (F.3) iff $e = 2, 2\epsilon_0$, in which case $f^{(0)}_{2\epsilon_0;(0);n} = (-4)^n \frac{(\epsilon_0+\frac{3}{2})_n}{n!(2)_n}$ and $f^{(0)}_{2;(0);n} = (-4)^n \frac{(\epsilon_0+\frac{3}{2})_n}{n!(2\epsilon_0)_n}$ corresponding to the generating functions given in (3.60).

Eq. (3.40) implies that $L^-_r \star T^{(0)}_{e;(0)}$ can vanish only if $2(\epsilon_0+1)e = e^2 + 4\epsilon_0$, that is $e = 2\epsilon_0$ or $e = 2$. Similarly, from (3.39) it follows that $M_{rs} \star T^{(0)}_{e;(0)}$ can vanish only if $e = \pm 2\epsilon_0$. Indeed, using (B.16) to derive the lemmas

$$\mathrm{ac}_{M_{0r}} T_{0(n)} \ = \ 2M_{0r}T_{0(n)} + \frac{1}{8}\frac{(n-1)n(n+1)(n+2\epsilon_0-1)}{(n+\epsilon_0-\frac{1}{2})(n+\epsilon_0+\frac{1}{2})} M_{0r}T_{0(n-2)} , \qquad \text{(F.22)}$$

$$\mathrm{ac}_{M_{rs}} T_{0(n)} \ = \ 2M_{rs}T_{0(n)} - \frac{1}{8}\frac{(n-1)n^2(n+1)}{(n+\epsilon_0-\frac{1}{2})(n+\epsilon_0+\frac{1}{2})} M_{rs}T_{0(n-2)} , \qquad \text{(F.23)}$$

one can then show that

$$L^-_r \star T^{(0)}_{2\epsilon_0;(0)} = L^-_r \star T^{(0)}_{2;(0)} \ = \ 0 , \qquad M_{rs} \star T^{(0)}_{2\epsilon_0;(0,0)} \ = \ 0 \ . \qquad \text{(F.24)}$$

G. Oscillator realizations

In this appendix we collect some basic properties of oscillator algebras, and some particular properties of the spinor-oscillator realization of 4D higher-spin representations.

G.1 On traces and projectors in oscillator algebras

We first discuss traces, inner products and projectors in the phase-space (two-sided) and Fock-space (one-sided) representations of oscillator algebras. To further illustrate ideas we also discuss generalized projectors and fermionic oscillators.

Phase-space trace and supertrace. The complexified Heisenberg algebra $u\star v - v\star u = 1$ generates the associative algebra of Weyl-ordered, i.e. symmetrized, functions $f(u,v)$ with product

$$f\star g = \int_{\mathbb{C}\times\mathbb{C}} \frac{d\xi d\bar{\xi} d\eta d\bar{\eta}}{\pi^2}\, e^{2i(\bar{\xi}\eta+\bar{\eta}\xi)} f(u+\xi, v+\bar{\xi}) g(u+i\eta, v-i\bar{\eta})\,, \tag{G.1}$$

where $d\xi d\bar{\xi} = 2d(\mathrm{Re}\xi)d(\mathrm{Im}\xi)$. The algebra admits the two inequivalent hermitian conjugations

$$u^\dagger = v\,,\quad v^\dagger = u\,,\qquad u^\ddagger = -v\,,\qquad v^\ddagger = -u\,, \tag{G.2}$$

and two associated inequivalent traces, namely the cyclic trace and the graded-cyclic supertrace

$$\mathrm{Tr}_+(f) = \int_{\mathbb{C}} \frac{du d\bar{u}}{2\pi} f(u,\bar{u})\,,\qquad \mathrm{Tr}_-(f) = \frac{f(0,0)}{2}\,, \tag{G.3}$$

obeying $\mathrm{Tr}_+(f\star g) = \mathrm{Tr}_+(fg) = \mathrm{Tr}_+(g\star f)$ up to boundary terms and $\mathrm{Tr}_-(f\star g) = (-1)^{\epsilon(f)\epsilon(g)}\mathrm{Tr}_-(g\star f)$ for functions f and g with definite parity defined by $f(-u,-v) = (-1)^{\epsilon(f)} f(u,v)$ idem g. The two traces are related as follows:

$$\mathrm{Tr}_\pm(f) = \mathrm{Tr}_\mp((-1)^N_\star \star f)\,,\qquad N = v\star u\,, \tag{G.4}$$

where we use the notation $x^A_\star = \exp_\star(A\ln x)$ with $\exp_\star A = \sum_{n=0}^\infty \frac{A^{\star n}}{n!}$ and $A^{\star n} = \underbrace{A\star\cdots\star A}_{n \text{ times}}$. Eq. (G.4) is a consequence of the Weyl-ordering formula

$$\exp_\star(\alpha w) = \frac{\exp(2w\tanh\frac{\alpha}{2})}{\cosh\frac{\alpha}{2}}\,,\qquad w = N + \frac{1}{2} = uv\,, \tag{G.5}$$

for $\alpha \in \mathbb{C}\backslash\{\pm i\pi, \pm 3i\pi, \dots\}$. This formula follows by acting with $\partial/\partial\alpha$ and using

$$w\star f(w) = \left(w - \frac{1}{4}\frac{\partial}{\partial w} - \frac{1}{4}w\frac{\partial^2}{\partial w^2}\right) f(w)\,. \tag{G.6}$$

Thus, setting $\exp_\star \alpha w = r(\alpha)\exp(s(\alpha)w)$, one finds $r' = -rs/4$ and $s' = 1 - s^2/4$ subject to $r(0) = 1$ and $s(0) = 0$, with the solution $r^{-1} = \cosh(\alpha/2)$ and $s = 2\tanh(\alpha/2)$. Eq. (G.4) then follows from

$$\exp_\star(i(\pi+\epsilon)N) \sim -i\frac{\exp\frac{2iuv}{\eta}}{\eta}\qquad \text{as } \eta = -\sin(\epsilon/2)\to 0\,, \tag{G.7}$$

which together with $Tr_+(f\star g) = \mathrm{Tr}_+(fg)$ implies

$$\lim_{\epsilon\to 0}\mathrm{Tr}_+(\exp_\star(i(\pi+\epsilon)N)\star f) = -i\lim_{\eta\to 0}\int_{\mathbb{C}}\frac{du d\bar{u}}{2\pi}\frac{\exp\frac{2iu\bar{u}}{\eta}}{\eta} f(u,\bar{u}) = \frac{f(0,0)}{2}\,. \tag{G.8}$$

Moreover, from $\exp_\star(\alpha N)\star\exp_\star(\beta N) = \exp_\star((\alpha+\beta)N)$, it follows that

$$\lim_{\epsilon\to 0}[\exp_\star(i(\pi+\epsilon)N)]^{\star 2} = \lim_{\epsilon\to 0}\exp_\star(2i(\pi+\epsilon)N) \tag{G.9}$$

$$= \lim_{\epsilon\to 0}\frac{\exp(2\tanh(i(\pi+\epsilon))N - i(\pi+\epsilon))}{\cosh i(\pi+\epsilon)} = 1\,, \tag{G.10}$$

in agreement with $(-1)^N_\star \star (-1)^N_\star = (-1)^{2N}_\star = 1$.

Fock-space inner products and projectors. The standard Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathbb{C} \otimes |n\rangle$, where $|n\rangle = \frac{v^n}{\sqrt{n!}}|0\rangle$ and $u|0\rangle = 0$, has two inequivalent inner products $I_\pm(|\Psi\rangle, |\Psi'\rangle) \equiv {}_\pm\langle\Psi|\Psi'\rangle$ defined by

$$I_\pm(\mu|m\rangle, \nu|n\rangle) = \bar{\mu}\nu(\pm 1)^m \delta_{mn}\,, \qquad \mu, \nu \in \mathbb{C}\,, \tag{G.11}$$

and related to the traces $\mathrm{Tr}_\pm$ by

$$I_\pm(|m\rangle, |n\rangle) = \mathrm{Tr}_\pm(P_{n,m})\,, \quad P_{n,m} = \frac{1}{\sqrt{m!n!}}\, v^n \star P_{0,0} \star u^m\,, \quad P_{0,0} = 2e^{-2w}\,. \tag{G.12}$$

One may identify $P_{n,m} \leftrightarrow |n\rangle_+\langle m| = |n\rangle\langle m|$, where $\langle\Psi| \equiv {}_+\langle\Psi|$, since the projector algebra $P_{m,n} \star P_{p,q} = \delta_{np} P_{m,q}$ follows from $P_{0,0} \star P_{0,0} = P_{0,0}$ and $u \star P_{0,0} = P_{0,0} \star v = 0$, and (G.12) follows from the cyclicity properties of $\mathrm{Tr}_\pm$ which imply $\mathrm{Tr}_\pm(P_{n,m}) = (\pm 1)^n \delta_{mn} \mathrm{Tr}_\pm(P_{0,0}) = (\pm 1)^n \delta_{mn}$ using $\mathrm{Tr}_\pm(P_{0,0}) = 1$. Moreover, it follows from (G.4) that ${}_\pm\langle\Psi|\Psi'\rangle = {}_\mp\langle\Psi|(-1)_\star^N|\Psi'\rangle$, which indeed induces the hermitian conjugation rules (G.2):

$$I_\pm(f|\Psi\rangle, |\Psi'\rangle) = I_\pm(|\Psi\rangle, g|\Psi'\rangle)\,, \qquad g = \begin{cases} f^\dagger \text{ for } I_+ \\ f^\ddagger \text{ for } I_- \end{cases}\,. \tag{G.13}$$

Next, writing the projectors on even and odd states as $P_{(\pm)} = \sum_{n=0}^{\infty} \frac{1}{2}(1 \pm (-1)^n)|n\rangle\langle n| = \frac{1}{2} : (e^N \pm e^{-N})|0\rangle\langle 0| :$ and using $1 = P_{(+)} + P_{(-)} =: e^N|0\rangle\langle 0| :$, yields the useful lemma:

$$(-1)_\star^N = : e^{-2N} :\,, \qquad |0\rangle\langle 0| = e^{-N} :\,, \qquad P_{(\pm)} = \frac{1}{2}(1 \pm : e^{-2N} :)\,. \tag{G.14}$$

To verify $\mathrm{Tr}_\pm(P_{n,n}) = (\pm 1)^n$, one may compute $P_{n,n} = |n\rangle\langle n| = 2(-1)^n e^{-2w} L_n(4w)$ by either direct evaluation of the $\star$-products in (G.12), or by using $: e^{au+bv} := e_\star^{au} \star e_\star^{bv} = e_\star^{au+bv+\frac{1}{2}ab} = e^{au+bv+\frac{1}{2}ab}$ followed by Fourier transformation:[39]

$$\begin{aligned} P_{n,n} = |n\rangle\langle n| &= \frac{1}{n!} : v^n e^{-vu} u^n : = \int \frac{dk d\bar{k}}{2\pi} : e^{-i(\bar{k}u + kv) - \bar{k}k} : L_n(\bar{k}k) \\ &= \sum_{p=0}^{n} \binom{n}{n-p} \frac{1}{p!} (\partial_u \partial_v)^p \int \frac{dk d\bar{k}}{2\pi} e^{-i(\bar{k}u+kv) - \frac{1}{2}\bar{k}k} = 2 \sum_{p=0}^{n} \binom{n}{n-p} (-2)^p e^{-2w} L_p(2w) \\ &= 2(-1)^n e^{-2w} L_n(4w)\,. \end{aligned} \tag{G.15}$$

Anti-Fock space and (anti-)automorphisms. The anti-Fock space is defined by

$$\mathcal{F}^- = \bigoplus_{n=0}^{\infty} \mathbb{C} \otimes |n\rangle^-\,, \qquad |n\rangle^- = \frac{u^n}{\sqrt{n!}}\,|0\rangle^-\,, \quad v|0\rangle^- = 0\,. \tag{G.16}$$

Its two inequivalent inner products are defined by

$${}^-_\pm\langle m|n\rangle^- = (\mp 1)^m \delta_{mn} = \mathrm{Tr}_\pm(P^-_{n,m})\,, \tag{G.17}$$

[39] The Laguerre polynomials $L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}(e^{-x} x^n) = \sum_{p=0}^{n} \binom{n}{n-p} \frac{(-1)^p}{p!} x^p$ obey $\sum_{n=0}^{\infty} (-z)^n L_n(2x) = \frac{e^{\frac{2xz}{1+z}}}{1+z} = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \binom{n}{n-p} (-2)^p L_p(x) z^n$.

with $P^-_{n,m} = |n\rangle^{--}\langle m| = \frac{1}{\sqrt{n!m!}}\, u^n \star P^-_{0,0} \star v^m$ and $P^-_{0,0} = 2e^{2w}$. It follows that $P^-_{m,n} \star P^-_{p,q} = (-1)^n \delta_{np} P^-_{m,q}$ and that

$$P^-_{n,n} = 2(-1)^n e^{2w} L_n(-4w)\ . \tag{G.18}$$

For uniformity we define $\mathcal{F}^+ = \mathcal{F}$, $|n\rangle^+ = |n\rangle$, ${}^+\langle n| = \langle n|$ and $P^+_{n,m} = P_{n,m}$, so that $w|n\rangle^\pm = \pm(n+\frac{1}{2})|n\rangle^\pm$ and ${}^\pm\langle n|w = \pm(n+\frac{1}{2}){}^\pm\langle n|$. The oscillator algebra has the automorphism π and anti-automorphism τ given by

$$\pi(f(u,v)) = f(iv,iu)\,, \qquad \tau(f(u,v)) \;=\; f(iu,iv)\,, \tag{G.19}$$

which exchange the $P^\pm$ projectors, viz. $\pi(P^\pm_{n,m}) = \tau(P^\pm_{n,m}) = i^{m+n} P^\mp_{m,n}$, and with compositions

$$\pi\circ\pi = \tau\circ\tau = \mathrm{Ad}_{(-1)^{N'}_\star}\,, \quad N' = \sum_{n=0}^{\infty}\left(\left(w-\frac{1}{2}\right)\star P^+_{n,n} + \left(w+\frac{1}{2}\right)\star P^-_{n,n}\right)\,, \tag{G.20}$$

where $\mathrm{Ad}_X(Y) = X \star Y \star X^{-1}$, and

$$R = \pi\circ\tau\,, \qquad R(f(u,v)) = \;f(-v,-u)\,, \qquad R\circ R \;=\; \mathrm{Id}\ . \tag{G.21}$$

The action of the discrete maps can be extended to the Fock spaces, such that

$$\pi\,:\,\mathcal{F}^\pm \oplus \mathcal{F}^{*\pm} \rightarrow \mathcal{F}^\mp \oplus \mathcal{F}^{*\mp}\,, \qquad \tau\;\; \mathcal{F}^\pm \rightarrow \mathcal{F}^{*\mp}\,, \qquad R\;:\;\mathcal{F}^\pm \rightarrow \mathcal{F}^{*\pm}\,, \tag{G.22}$$

upon defining

$$\pi(|0\rangle^\pm) = |0\rangle^\mp\,, \qquad \pi({}^\pm\langle 0|) \;=\; {}^\mp\langle 0|\,, \tag{G.23}$$

$$\tau(|0\rangle^\pm) = {}^\mp\langle 0|\,, \qquad \tau({}^\pm\langle 0|) \;=\; |0\rangle^\mp\,, \tag{G.24}$$

and $\mathrm{Ad}_X(|\Psi\rangle) = X|\Psi\rangle$ and $\mathrm{Ad}_X(\langle\Psi|) = \langle\Psi|X^{-1}$, such that (G.20) remains valid. It follows that

$$R(|0\rangle^\pm) = {}^\pm\langle 0|\,, \qquad R({}^\pm\langle 0|) \;=\; |0\rangle^\pm\,, \tag{G.25}$$

that is, $R(\cdot) \leftrightarrow (\cdot)^\ddagger$ in the real basis $\{|n\rangle^\pm, {}^\pm\langle n|\}$ and where $\ddagger$ is defined in (G.2). In section G.2 the discrete maps acting on pseudo-real $\mathfrak{su}(2)$-doublets are defined analogously with conventiones preserving SU(2) quantum numbers.

Generalized projectors and their composition. To illustrate the distinction between operators in Fock spaces and more general classes of phase-space functions, we consider analytic functions $M^C_\kappa(w)$ obeying

$$(w-\kappa)\star M^C_\kappa \;=\; 0\,, \qquad \kappa\in\mathbb{C}\ , \tag{G.26}$$

where C indicize a basis of linearly independent solutions. One may consider

$$M^C_\kappa \;=\mathcal{N}^C_\kappa \oint_C \frac{d\alpha}{2\pi i} g^{(\kappa)}(\alpha) \;=\; \mathcal{N}^C_\kappa \oint_\Gamma \frac{ds\; g^{(\kappa)}(\alpha(s))}{2\pi i(1-\frac{s^2}{4})} \;=\; \mathcal{N}^C_\kappa \oint_{\Gamma'} \frac{d\lambda\; g^{(\kappa)}(\alpha(\lambda))}{2\pi i(1+\lambda)}\,, \tag{G.27}$$

where C, $\Gamma(C)$ and $\Gamma'(C)$ are contours in the α, s and λ planes; $g^{(\kappa)}(\alpha) = e_\star^{\alpha(w-\kappa)}$ with Weyl-ordered and normal-ordered forms

$$g^{(\kappa)}(\alpha(s)) = \left(1+\frac{s}{2}\right)^{\frac{1}{2}-\kappa}\left(1-\frac{s}{2}\right)^{\frac{1}{2}+\kappa} e^{sw}\ , \qquad s = 2\tanh\frac{\alpha}{2}\ , \tag{G.28}$$

$$g^{(\kappa)}(\alpha(\lambda)) = (1+\lambda)^{\frac{1}{2}-\kappa} : e^{\lambda w} :\ , \qquad \lambda = \frac{s}{1-\frac{s}{2}} \ = \ e^{\alpha}-1\ ; \tag{G.29}$$

and $\mathcal{N}_\kappa^C \in \mathbb{C}$ are normalizations such that $M_\kappa^C(0) = 1$. In view of (G.6), eq. (G.26) holds if $[g^{(\kappa)}(\alpha(s))]|_{\partial C} = 0$. In particular, if $C = C_{i\pi}$ is a small closed contour encircling $i\pi$ clockwise, then its image $\Gamma_{[2,-2]} \equiv s(C_{i\pi})$ is a large contour encircling $[-2,2]$ counterclockwise.[40] Enlarging $C_{i\pi}$ to the "box" $\{i\epsilon+x : -L \leqslant x \leqslant L\} \cup \{L+ix : \epsilon \leqslant x \leqslant 2\pi-\epsilon\} \cup \{i(2\pi-\epsilon)-x : -L \leqslant x \leqslant L\} \cup \{-L+i(2\pi-x) : \epsilon \leqslant x \leqslant 2\pi-\epsilon\}$, its image $\Gamma_{[-2,2]}$ shrinks to a "dogbone" containing $[-2,2]$. For $\kappa = \pm(n+1/2)$, there is no branch cut, $\oint_{\Gamma_{[-2,2]}} \to \oint_{\pm 2}$, and one finds

$$M^{C_{i\pi}}_{\pm(n+\frac{1}{2})} = 2(\mp 1)^n e^{\mp 2w} L_n(\pm 4w) \ = \ P^{\pm}_{n,n} \quad \text{for } \mathcal{N}^{C_{i\pi}}_{\pm(n+\frac{1}{2})} = (\pm 1)^{n+1}\ . \tag{G.30}$$

Their $\star$-products can be computed using the composition rule

$$g^{(\kappa)}(\alpha) \star g^{(\kappa)}(\alpha') = g^{(\kappa)}(\alpha+\alpha') \ = \ g^{(\kappa)}(\alpha(s'')) \ = \ g^{(\kappa)}(\alpha(\lambda''))\ , \tag{G.31}$$

with $s'' = \frac{s+s'}{1+\frac{ss'}{4}}$ and $\lambda'' = \lambda+\lambda'+\lambda\lambda'$. This rule, which is equivalent to $e^{sw} \star e^{s'w} = \frac{1}{1+\frac{ss'}{4}} e^{s''w}$ and $: e^{\lambda w} : \star : e^{\lambda' w} =: e^{\lambda'' w} :$, holds by analytical continuation for all s and s' such that $ss' \neq -4$. A change of variables and analytical contour deformation then yield

$$\begin{aligned}(M^{C_{i\pi}}_{\pm(n+\frac{1}{2})})^{\star 2} &= \left[\mathcal{N}^{C_{i\pi}}_{\pm(n+\frac{1}{2})} \oint_{\mp 2} \frac{ds}{2\pi i(1-\frac{s^2}{4})}\right] M^{C_{i\pi}}_{\pm(n+\frac{1}{2})} \\ &= \pm \mathcal{N}^{C_{i\pi}}_{\pm(n+\frac{1}{2})} M^{C_{i\pi}}_{\pm(n+\frac{1}{2})} = (-1)^n M^{C_{i\pi}}_{\pm(n+\frac{1}{2})}\ ,\end{aligned} \tag{G.32}$$

in agreement with $(P^{\pm}_{n,n})^{\star 2} = (-1)^n P^{\pm}_{n,n}$. For $\kappa \notin (\mathbb{Z}+\frac{1}{2})$, the branch cut along $[-2,2]$ prevents $\Gamma_{[-2,2]}$ from collapsing, and $(M_\kappa^{C_{i\pi}})^{\star 2}$ is computed with s and s' lying on large $\Gamma_{[-2,2]}$-contours; the change of variables now yields[41]

$$M_\kappa^{\star 2} = \left[\mathcal{N}_\kappa^{C_{i\pi}} \oint_{\Gamma_{[-2,2]}} \frac{ds}{2\pi i(1-\frac{s^2}{4})}\right] M_\kappa^{C_{i\pi}} \ = \ 0 \text{ for } \kappa \notin (\mathbb{Z}+\tfrac{1}{2})\ . \tag{G.33}$$

Fermionic oscillators. In the case of fermionic oscillators, there is less distinction between Fock-space and phase-space formulations. The complexified Clifford algebra

[40] Other interesting choices of closed contours are $C = i[-\pi,\pi]$ and $C = \mathbb{R}$ leading to integrals over U(1) and $GL(1;\mathbb{R})$, respectively.

[41] The functions $M_0^{C_{i\pi}}$ are related to the dressing functions M appearing in the perturbative weak-field expansion of the oscillator formulations of higher-spin gauge theories [48, 49, 21, 40], which are the analytical solutions to $K \star M = 0$ for K belonging to the "internal" gauge group of the oscillator algebra, which is U(1), SU(2) and Sp(2), respectively, for 5D spinor, 7D spinor and D-dimensional vector oscillators.

$\{\gamma,\delta\}_\star = 1$ has isomorphic Fock and anti-Fock spaces, generated by $|0\rangle$ and $|1\rangle = \delta|0\rangle$ obeying $\gamma|0\rangle = 0$. The inequivalent inner products $I_\pm$ are related by ${}_\pm\langle\Psi|\Psi'\rangle = {}_\mp\langle\Psi|(-1)^F_\star \star \Psi'\rangle$ with $F \equiv \delta\star\gamma$, and induce the hermitian conjugation rules

$$\gamma^\dagger = \delta\,, \quad \delta^\dagger \;=\; \gamma\,, \qquad \gamma^\ddagger \;=\; -\delta\,, \quad \delta^\ddagger \;=\; -\gamma\,, \tag{G.34}$$

via $I_+(f\star|\Psi\rangle,|\Psi'\rangle) = I_+(|\Psi\rangle, f^\dagger\star|\Psi'\rangle)$ and $I_-(f\star|\Psi\rangle,|\Psi'\rangle) = I_-(|\Psi\rangle, f^\ddagger\star|\Psi'\rangle)$. The phase-space formulation uses the Weyl-ordered product $\gamma\star\delta = \gamma\delta + 1/2$, with $\gamma\delta = [\gamma,\delta]_\star/2 = -\delta\gamma$, and the trace operations

$$\mathrm{Tr}_+(f) = 2\,f(0,0)\,, \qquad \mathrm{Tr}_-(f) \;=\; -\int d\gamma d\bar\gamma\, f(\gamma,\bar\gamma)\,, \tag{G.35}$$

where $f = f(\gamma,\delta)$ is Weyl-ordered and $\int d\gamma d\bar\gamma\;\bar\gamma\gamma = 1$, and we note the interchanged role of Tr_+ and Tr_- in comparison to bosons. For example, in terms of the projectors $P_0 = |0\rangle\langle 0| = 1 - \delta\star\gamma = \frac{1}{2}(1-2\delta\gamma)$ and $P_1 = |1\rangle\langle 1| = \delta\star\gamma = \frac{1}{2}(1+2\delta\gamma)$ one has $\mathrm{Tr}_+(P_0) = \mathrm{Tr}_+(P_1) = 1$ and $\mathrm{Tr}_-(P_0) = -\mathrm{Tr}_-(P_1) = 1$. One can show that

$$\mathrm{Tr}_\pm((-1)^F_\star \star f) \;=\; \mathrm{Tr}_\mp(f)\;. \tag{G.36}$$

To this end, one uses $\int d\gamma d\bar\gamma\, f\star g = \int d\gamma d\bar\gamma\, fg$, and $F = P_1$ which implies that

$$(-1)^F_\star = \exp_\star(i\pi F) = 1 + \sum_{n=1}^{\infty}\frac{(i\pi)^n}{n!}P_1 = 1 + (e^{i\pi}-1)P_1 = 1 - 2P_1 = -2\bar\gamma\gamma\,, \tag{G.37}$$

which indeed implies $(-1)^F_\star\star(-1)^F_\star = (-2\bar\gamma\gamma)\star(-2\bar\gamma\gamma) = 1$. Finally, using $\bar\gamma\gamma = \delta(\bar\gamma)\delta(\gamma)$, one finds

$$\mathrm{Tr}_-((-1)^F_\star \star f) = -\int d\gamma d\bar\gamma\,(-2\bar\gamma\gamma)f(\gamma,\bar\gamma) \;=\; 2f(0,0) \;=\; \mathrm{Tr}_+(f)\;. \tag{G.38}$$

G.2 4D Spinor-oscillator realizations

In $D = 4$ the algebra $\mathcal{A}$ is isomorphic to the space of Weyl-ordered even arbitrary polynomials [33, 51]

$$f(y,\bar y) = \sum_{\substack{n,\,m\\ n+m \text{ even}}} f^{\alpha(n),\dot\alpha(m)}T_{\alpha(n),\dot\alpha(m)}\,, \quad T_{\alpha(n),\dot\alpha(m)} \;=\; \frac{1}{n!m!}y_{\alpha_1}\cdots y_{\alpha_n}\bar y_{\dot\alpha_1}\cdots\bar y_{\dot\alpha_m}\,, \tag{G.39}$$

in an $\mathfrak{sl}(2;\mathbb{C})_L \oplus \mathfrak{sl}(2;\mathbb{C})_R$-quartet $(y_\alpha, \bar y_{\dot\alpha})$ obeying

$$y_\alpha\star y_\beta = y_\alpha y_\beta + i\epsilon_{\alpha\beta}\,, \quad \bar y_{\dot\alpha}\star\bar y_{\dot\beta} = \bar y_{\dot\alpha}\bar y_{\dot\beta} + i\epsilon_{\dot\alpha\dot\beta}\,, \quad y_\alpha\star\bar y_{\dot\beta} = \bar y_{\dot\beta}\star y_\alpha = y_\alpha\bar y_{\dot\beta}\,, \tag{G.40}$$

where juxtaposition denotes Weyl-ordered products and $T_{\alpha(n),\dot\alpha(m)}$ has $\mathfrak{sl}(2;\mathbb{C})_L\oplus\mathfrak{sl}(2;\mathbb{C})_R$ spin $(j_L, j_R) = \frac{1}{2}(n,m)$ and Lorentz spin $(s_1,s_2) = (\frac{n+m}{2}, \frac{|n-m|}{2})$. The $\star$-product reads[42]

$$f(y,\bar y)\star g(y,\bar y) \;=\; \int\frac{d^2\xi d^2\eta d^2\bar\xi d^2\bar\eta}{(2\pi)^4}e^{i\eta^\alpha\xi_\alpha + i\bar\eta^{\dot\alpha}\bar\xi_{\dot\alpha}}f(y+\xi,\bar y+\bar\xi)g(y+\eta,\bar y+\bar\eta)\,. \tag{G.41}$$

[42] The composition rules in more general classes of functions require separate definitions; the case of the oscillator realization of the compact basis elements defined in (3.1) shall be discussed in [36].

JHEP10(2008)022

The anti-automorphism τ and automorphism π can be taken to be

$$\tau(f(y,\bar{y})) = f(iy,i\bar{y})\,, \quad \pi(f(y,\bar{y})) = f(-y,\bar{y})\,, \qquad \bar{\pi}(f(y,\bar{y})) = f(y,-\bar{y})\ . \quad \text{(G.42)}$$

The realization of the ℓth levels of the bosonic higher-spin algebra $\mathfrak{ho}(5;\mathbb{C})$ and its twisted-adjoint representation $\mathcal{T}(5;\mathbb{C})$ read

$$Q_\ell = \sum_{n+m=4\ell+2} Q_{\alpha_1\dots\alpha_n\dot{\alpha}_1\dots\dot{\alpha}_m} T^{\alpha(n),\dot{\alpha}(m)}\,, \qquad \ell = -\frac{1}{2},0,\frac{1}{2},1,\dots\,, \quad \text{(G.43)}$$

$$S_\ell = \sum_{|n-m|=4\ell} S^{\alpha(n),\dot{\alpha}(m)} T_{\alpha(n),\dot{\alpha}(m)}\ , \qquad \ell = -1,-\frac{1}{2},0,\frac{1}{2},1,\dots\,, \quad \text{(G.44)}$$

and the minimal algebra is obtained by truncating to integer ℓ.

The hermitian conjugation is defined in various signatures by [37]

$$(f)^\dagger = \iota((f)^{\dagger_{\rm osc}})\,, \quad \text{(G.45)}$$

where:

(i) $\dagger_{\rm osc}$ acts on the oscillators in different signature of the real form of $\mathfrak{m}$ as follows:

$$\begin{aligned} \mathfrak{su}(2)_L\oplus\mathfrak{su}(2)_R: &\quad (y^\alpha)^{\dagger_{\rm osc}} = y^\dagger_\alpha\,, \quad (\bar{y}^{\dot{\alpha}})^{\dagger_{\rm osc}} = \bar{y}^\dagger_{\dot{\alpha}}\,, & \text{(G.46)}\\ \mathfrak{sl}(2;\mathbb{C})_{\rm diag}: &\quad (y^\alpha)^{\dagger_{\rm osc}} = \bar{y}^{\dot{\alpha}}\,, & \text{(G.47)}\\ \mathfrak{sp}(2;\mathbb{R})_L\oplus\mathfrak{sp}(2;\mathbb{R})_R: &\quad (y^\alpha)^{\dagger_{\rm osc}} = y^\alpha\,, \quad (\bar{y}^{\dot{\alpha}})^{\dagger_{\rm osc}} = \bar{y}^{\dot{\alpha}}\ ; & \text{(G.48)} \end{aligned}$$

and

(ii) ι is an oscillator-algebra isomorphism that acts in different signatures of the real forms of $\mathfrak{g}$ and $\mathfrak{m}$ as follows:

$$\begin{aligned} \mathfrak{so}(5)\supset\mathfrak{so}(4): &\quad \iota = \rho\,, & \text{(G.49)}\\ \mathfrak{so}(1,4)\supset\mathfrak{so}(4): &\quad \iota = \pi\rho\,, & \text{(G.50)}\\ \mathfrak{so}(1,4)\supset\mathfrak{so}(1,3): &\quad \iota = \pi\,, & \text{(G.51)}\\ \mathfrak{so}(2,3)\supset\mathfrak{so}(1,3): &\quad \iota = \mathrm{Id}\,, & \text{(G.52)}\\ \mathfrak{so}(2,3)\supset\mathfrak{so}(2,2): &\quad \iota = \mathrm{Id}\,, & \text{(G.53)} \end{aligned}$$

where in Euclidean signature the SU(2) doublets are pseudo real (i.e. $(y_\alpha)^{\dagger_{\rm osc}} = -y^{\dagger\alpha}$ idem $\bar{y}_{\dot{\alpha}}$), and $(y_\alpha,\bar{y}_{\dot{\alpha}})$ and $(y^\dagger_\alpha,\bar{y}^\dagger_{\dot{\alpha}})$ generate equivalent oscillator algebras with isomorphism

$$\rho(f(y^\dagger_\alpha,\bar{y}^\dagger_\alpha)) = f(y_\alpha,\bar{y}_\alpha)\ . \quad \text{(G.54)}$$

The hermitian conjugation obeys $((f)^\dagger)^\dagger = \iota((\iota((f)^{\dagger_{\rm osc}}))^{\dagger_{\rm osc}}) = f$, which relies on $\pi\bar{\pi}(f) = f$ for the real form $\mathfrak{so}(1,4)$. Thus, the real forms of the generators M_{AB} obey $(M^{\mathbb{R}}_{AB})^\dagger =$

$\iota((M^{\mathbb{R}}_{AB})^{\dagger_{\rm osc}}) = M^{\mathbb{R}}_{AB}$, and have the oscillator realizations[43]

$$M_{ab} = -\frac{1}{8}\left[\,(\sigma_{ab})^{\alpha\beta}y_\alpha y_\beta + (\bar\sigma_{ab})^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\bar y_{\dot\beta}\,\right]\,,\qquad P_a \;=\; \frac{\sqrt{\lambda}}{4}(\sigma_a)^{\alpha\dot\beta}y_\alpha\bar y_{\dot\beta}\;. \tag{G.58}$$

The Fock-space and anti-Fock-space submodules in $\mathcal{A}$ are exhibited by going from $(y_\alpha,\bar y_{\dot\alpha})$ to a $\mathfrak{u}(2)$-covariant basis $(a_i,\bar a^i)$, $i=1,2$ [52], obeying

$$[a_i,\bar a^j]_\star \;=\; \delta_i^j\,,\qquad [a_i,a_j]_\star \;=\; [\bar a^i,\bar a^j]_\star \;=\; 0\;, \tag{G.59}$$

after which the $\mathfrak{so}(5;\mathbb{C})$ generators can be expressed as

$$E = \frac{1}{2}(\bar a^i a_i + 1)\,,\qquad\qquad M_{rs} \;=\; \frac{i}{2}(\sigma_{rs})_i{}^j\bar a^i a_j\,, \tag{G.60}$$

$$L^+_r = \frac{i}{2}(\sigma_r)_{ij}\bar a^i\bar a^j\,,\qquad\qquad L^-_r = \frac{i}{2}(\sigma_r)^{ij}a_i a_j\;. \tag{G.61}$$

The Fock and anti-Fock spaces

$$\mathcal{F}^\pm = \bigoplus_{n=0}^{\infty}\mathbb{C}\otimes|n\rangle^\pm\,,\qquad |n\rangle^+ \;=\; \bar a^{i_1}\cdots\bar a^{i_n}|0\rangle\;,\quad |n\rangle^- \;=\; a^{i_1}\cdots a^{i_n}|0\rangle^-\,, \tag{G.62}$$

can be identified as the $\mathfrak{osp}(4|1)$ supersingleton

$$\mathcal{F}^\pm \;=\; \mathfrak{D}^\pm_0\oplus\mathfrak{D}^\pm_{1/2}\,, \tag{G.63}$$

where $\mathfrak{D}^\pm_0$ and $\mathfrak{D}^\pm_{1/2}$ consist of the even and odd states, respectively, and $|\frac{1}{2};(0)\rangle = |0\rangle$ and $|1;(\frac{1}{2})\rangle^i = \;\bar a^i|0\rangle$. The lowest-weight states of the composite-massless scalars now read[44]

$$|1;(0)\rangle_{12} \;=\; |\frac{1}{2};(0)\rangle_1|\frac{1}{2};(0)\rangle_2 \;=\; |0\rangle_1|0\rangle_2\,, \tag{G.64}$$

$$|2;(0)\rangle \;=\; |1;\left(\frac{1}{2}\right)\rangle^i_1|1;\left(\frac{1}{2}\right)\rangle_{2;i} \;=\; -y|1;(0)\rangle_{12}\,, \tag{G.65}$$

[43] Raising and lowering of two-component indices follow the convention $y^\alpha = \epsilon^{\alpha\beta}y_\beta$ and $y_\alpha = y^\beta\epsilon_{\beta\alpha}$ where $\epsilon^{\alpha\beta}\epsilon_{\gamma\delta} = 2\delta^{\alpha\beta}_{\gamma\delta}$ idem $\bar y_{\dot\alpha}$ and $\epsilon_{\dot\alpha\dot\beta}$. The van der Waerden symbols $(\sigma^a)_{\alpha\dot\alpha} = (\bar\sigma^a)_{\dot\alpha\alpha}$ obey

$$(\sigma^a)_\alpha{}^{\dot\alpha}(\bar\sigma^b)_{\dot\alpha}{}^\beta \;=\; \eta^{ab}\delta^\beta_\alpha \;+\; (\sigma^{ab})_\alpha{}^\beta\,,\qquad (\bar\sigma^a)_{\dot\alpha}{}^\alpha(\sigma^b)_\alpha{}^{\dot\beta} \;=\; \eta^{ab}\delta^{\dot\beta}_{\dot\alpha} \;+\; (\bar\sigma^{ab})_{\dot\alpha}{}^{\dot\beta}\,, \tag{G.55}$$

$$\frac{1}{2}\epsilon_{abcd}(\sigma^{cd})_{\alpha\beta} \;=\; \begin{cases}(\sigma_{ab})_{\alpha\beta}\,, & (4,0)\text{ and }(2,2)\text{ signature}\,,\\ i(\sigma_{ab})_{\alpha\beta}\,, & (3,1)\text{ signature}\,,\end{cases} \tag{G.56}$$

and the reality conditions

$$\Big(\epsilon_{\alpha\beta},(\sigma^a)_{\alpha\dot\beta},(\sigma^{ab}_{\alpha\beta})\Big)^\dagger \;=\; \begin{cases}\Big(\epsilon^{\alpha\beta},-(\bar\sigma^a)^{\dot\beta\alpha},(\sigma^{ab})^{\alpha\beta}\Big) & \text{for SU}(2)\,,\\ \Big(\epsilon_{\dot\alpha\dot\beta},(\bar\sigma^a)_{\dot\alpha\beta},(\bar\sigma^{ab})_{\dot\alpha\dot\beta}\Big) & \text{for SL}(2,\mathbb{C})\,,\\ \Big(\epsilon_{\alpha\beta},(\bar\sigma^a)_{\dot\beta\alpha},(\sigma^{ab})_{\alpha\beta}\Big) & \text{for Sp}(2)\;.\end{cases} \tag{G.57}$$

One may verify the conventions using $\epsilon = i\sigma^2$ and for SU(2), SL(2; $\mathbb{C}$) and SL(2; $\mathbb{R}$), respectively, $[\sigma^a,\bar\sigma^a]$ given by $[(i,\sigma^i),(-i,\sigma^i)]$, $[(-i\sigma^2,-i\sigma^i\sigma^2),(-i\sigma^2,i\sigma^2\sigma^i)]$ and $[(1,\tilde\sigma^i)),(-1,\tilde\sigma^i)]$ where $\tilde\sigma^i = (\sigma^1,i\sigma^2,\sigma^3)$.

[44] The conformal algebra $\mathfrak{so}(4,2)$ can be realized in $\mathcal{F}(1)\otimes\mathcal{F}(2)$ as $M_{AB} = \frac{1}{8}\sum_{\xi=1,2}\bar Y(\xi)\Gamma_{AB}Y(\xi)$, that act in a tensorial split, and $R_a = \frac{1}{4}\bar Y(1)\Gamma_A Y(2)$, that act in a non-tensorial split, where $Y_\alpha(\xi) = \bar Y_\alpha(\xi)$ are Majorana spinors obeying $Y_\alpha(\xi)\star Y_\beta(\eta) = Y_\alpha(\xi)Y_\beta(\eta) + i\delta_{\xi\eta}C_{\alpha\beta}$. The conformal generators can be re-written as $M_{AB} = \frac{1}{4}\bar u\Gamma_{AB}u$ and $R_A = -\frac{i}{2}\bar u\Gamma_A u$ where $u_\alpha = \frac{1}{\sqrt{2}}(Y_\alpha(1)+Y_\alpha(2))$ and $\bar u_\alpha = \frac{1}{\sqrt{2}}(Y_\alpha(1)-Y_\alpha(2))$ are Weyl spinors obeying $u_\alpha\star\bar u_\beta = u_\alpha\bar u_\beta + iC_{\alpha\beta}$.

JHEP10(2008)022

where

$$y \equiv \sqrt{x^+} = \sqrt{(L^+(1)+L^+(2))^2} = \bar{a}_i(1)\bar{a}^i(2) \ . \tag{G.66}$$

To make contact with (3.154) and (3.161) we define the reflection by $R(f\star g)=R(g)\star R(f)$ and

$$R(|0\rangle^\pm) = {}^\pm\langle 0| \,, \qquad R(\bar{a}^i) = ia^i \,, \quad R(a^i) = i\bar{a}^i \ . \tag{G.67}$$

It follows that $R(|n\rangle) = i^n\langle 0|a^{i_1}\cdots a^{i_n}$ and $R(|n\rangle^-) = i^{n-}\langle 0|\bar{a}^{i_1}\cdots\bar{a}^{i_n}$. We also define

$$\pi(|0\rangle^\pm) = |0\rangle^\mp \,, \qquad \tau(|0\rangle^\pm) = {}^\mp\langle 0| \,, \qquad R = \pi\circ\tau \ . \tag{G.68}$$

Using $z = 2L_r^+L_r^- =: N^2 :$ where $N = \bar{a}^ia_i$ and $:\bar{a}^ia^j := \bar{a}^ia^j$, we have $1\!\!1_{\mathcal{F}} = 1\!\!1_{\mathfrak{D}_0} + 1\!\!1_{\mathfrak{D}_{\frac{1}{2}}}$ with

$$1\!\!1_{\mathfrak{D}_0} = \sum_{\text{n even}} |n\rangle\langle n| = \sum_{\text{n even}} \frac{1}{n!}\bar{a}^{i_1}\ldots\bar{a}^{i_n}|0\rangle\langle 0|a_{i_1}\ldots a_{i_n} = : \cosh\sqrt{z}|0\rangle\langle 0| : , \tag{G.69}$$

$$1\!\!1_{\mathfrak{D}_{1/2}} = \sum_{\text{n odd}} |n\rangle\langle n| = \sum_{\text{n odd}} \frac{1}{n!}\bar{a}^{i_1}\ldots\bar{a}^{i_n}|0\rangle\langle 0|a_{i_1}\ldots a_{i_n} = : \sinh\sqrt{z}|0\rangle\langle 0| : \ . \tag{G.70}$$

Thus $1\!\!1_{\mathcal{F}} =: e^{\sqrt{z}}|0\rangle\langle 0|$: which implies $|0\rangle\langle 0| =: e^{-N}$:, and hence $1\!\!1_{\mathfrak{D}_0} = \frac{1}{2}(1+\Gamma)$ and $1\!\!1_{\mathfrak{D}_{1/2}} = \frac{1}{2}(1-\Gamma)$ with $\Gamma =: e^{-2N}$: obeying $\Gamma\star\Gamma = 1$. Applying $(R_2)^{-1}$ to $1\!\!1_{\mathcal{F}}$ thus yields the superreflector

$$|1\!\!1_{\mathcal{F}}\rangle_{12} = e^{iy}|0\rangle_1|0\rangle_2 = \cos y|1;(0)\rangle_{12} - i\frac{\sin y}{y}|2;(0)\rangle_{12} \ , \tag{G.71}$$

where we have used (G.64) and (G.65). The two composite trace operations on $\mathcal{F}^+$ read

$$\mathrm{Tr}_\pm(f) = \mathrm{Tr}_{\mathfrak{D}_0}(f) \pm \mathrm{Tr}_{\mathfrak{D}_{1/2}}(f) \,, \tag{G.72}$$

where $\mathrm{Tr}_{\mathfrak{D}_0}(f) = \sum_{n\text{ even}}\langle n|f|n\rangle$ and $\mathrm{Tr}_{\mathfrak{D}_{1/2}}(f) = \sum_{n\text{ odd}}\langle n|f|n\rangle$. As shown in appendix G.1, the odd composite trace coincides with the supertrace [33], that is[45]

$$\mathrm{Tr}_-(f) = \mathrm{Tr}_+((-1)^N_\star\star f) = \frac{1}{4}f(0,0) = \frac{1}{4}\mathrm{Str}(f) \ . \tag{G.73}$$

Thus, the appearance of the spinor singleton in $D=4$ as an extra solution to $V\approx 0$ leads to the identification (2.43) of the non-composite trace Tr defined in (2.42) with the supertrace.

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JHEP10(2008)022

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